Warwick Mathematics Exchange

MA3H6

Algebraic Topology

2024, February 3rd

Desync, aka The Big Ree

Contents

Introduction

Topology is the branch of mathematics concerned with continuity and connectedness, and properties of spaces that are invariant under continuous deformations.

The (fundamental) homotopy groups discussed in *MA3F1: Introduction to Topology* are a powerful invariant for some topological spaces, but they are unable to distinguish topological spaces in higher dimensions, and the higher dimensional analogues of nth homotopy groups become incredibly difficult to calculate – the homotopy groups of just higher dimensional spheres are unknown. Instead, we study *homology* groups, which are slightly less powerful, but generally much easier to compute. But in exchange, we require significantly more preamble before we may develop much theory.

Disclaimer: I make *absolutely no guarantee* that this document is complete nor without error. In particular, any content covered exclusively in lectures (if any) will not be recorded here. This document was written during the 2023 academic year, so any changes in the course since then may not be accurately reflected.

Notes on formatting

New terminology will be introduced in *italics* when used for the first time. Named theorems will also be introduced in *italics*. Important points will be **bold**. Common mistakes will be underlined. The latter two classifications are under my interpretation. YMMV.

Content not taught in the course will be outlined in the margins like this. Anything outlined like this is not examinable, but has been included as it may be helpful to know alternative methods to solve problems.

The table of contents above, and any inline references are all hyperlinked for your convenience.

History

First Edition: 2024-01-27*
Current Edition: 2024-02-03

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Please send me a PM on Discord @Desync#6290, a message in the WMX server, or an email to Warwick.Mathematics.Exchange@gmail.com for any corrections. (If this document somehow manages to persist for more than a few years, these contact details might be out of date, depending on the maintainers. Please check the most recently updated version you can find.)

If you found this guide helpful and want to support me, you can buy me a coffee!

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*Storing dates in big-endian format is clearly the superior option, as sorting dates lexicographically will also sort dates chronologically, which is a property that little and middle-endian date formats do not share. See ISO-8601 for more details. This footnote was made by the computer science gang.

1 Preliminary Concepts

1.1 Note on Notation

In general, we will take the word "map" to mean a *continuous* function, and "space" to mean a *topological* space.

We will write $X \twoheadrightarrow Y$ to denote an injective map (more generally, a monomorphism) and $X \rightarrowtail Y$ for a surjective map (more generally, an epimorphism). In some texts, $X \hookrightarrow Y$ is used for monomorphisms, but here we reserve this symbol exclusively for inclusion maps. We use no special notation for projection maps.

We write $f^{-1}[X]$ for the preimage of a *set* X under a function f.

1.2 Common topological spaces

We list some standard topological spaces:

- The unit interval I is the subspace $I \coloneqq [0,1] \subset \mathbb{R}$.
- The (closed) n-disc D^n is the subspace of $\mathbb{R}^n$ defined by

$$D^n \coloneqq \left\{ \mathbf{x} \in \mathbb{R}^n : \sum_{i=1}^{n} x_i^2 \leq 1 \right\}$$

- The n-sphere S^n is the boundary of the $(n+1)$-disc

$$S^n \coloneqq \partial D^n = \left\{ \mathbf{x} \in \mathbb{R}^n : \sum_{i=1}^{n} x_i^2 = 1 \right\}$$

- The n-cube I^n is the n-fold product of the unit interval I:

$$I^n \coloneqq \prod_{i=1}^{n} I \cong \left\{ \mathbf{x} \in \mathbb{R}^n : 0 \leq x_i \leq 1 \right\}$$

- The n-torus T^n is the n-fold product of the 1-sphere:

$$T^n \coloneqq \prod_{i=1}^{n} S^1 \cong \mathbb{R}^n / \mathbb{Z}^n$$

1.3 Homotopies

If $f,g : X \to Y$ are maps between topological spaces X and Y, then a *homotopy* from f to g is a map $H : X \times I \to Y$ such that $H(-,0) = f$ and $H(-,1) = g$. This second variable is commonly denoted by t and called the *time* parameter. Intuitively, a homotopy parametrises (in the second variable) a continuous deformation of f to g.

If there exists a homotopy between f and g, we say they are *homotopic* and write $f \simeq g$ (or occasionally $f \simeq_H g$, if the particular homotopy is relevant).

Two topological spaces X and Y are *homotopy equivalent* if there exist maps $f : X \to Y$ and $g : Y \to X$ such that $g \circ f \cong \mathrm{id}_X$ and $f \circ g \cong \mathrm{id}_Y$.

A space is *contractible* if it is homotopy equivalent to a point.

1.4 Pairs

A *pair* (X,A) consists of a topological space X and a subspace $A \subseteq X$. We denote the interior of A by A°, the closure of A by $\overline{A}$, and the boundary of A by $\partial A = \overline{A} \setminus A^\circ$. When $A = \{x\}$ is a single point, we instead write (X,x), and call the pair a *pointed space* (we sometimes call X alone a pointed space with *basepoint* x).

A *map of pairs* $f : (X,A) \to (Y,B)$ is a continuous function $f : X \to Y$ such that $f(A) \subseteq B$. If A and B are points, then f is a *pointed* or *based* map. If $f,g : (X,A) \to (Y,B)$ are maps such that $f\big|_A = g\big|_A$, then a *homotopy relative to* A from f to g is a homotopy $H : X \times I \to Y$ such that $H(x,t) = f(x) = g(x)$ for all $x \in A$ and $t \in I$. That is, a homotopy relative to A is a homotopy that is constant over A. Again, if A and B are points, then H is a *pointed homotopy.*

Given a pair (X,A), a *retraction* is a map $r : X \to X$ such that $r(X) = A$ and $r\big|_A = \mathrm{id}_A$. That is, a retraction is a (necessarily surjective) mapping from a space X onto a subspace A that preserves all points within that subspace. For instance, any non-empty space retracts to a point in the obvious way (just take the constant map). If a retraction exists, then A is a *retract* of X.

A *deformation retract* is a homotopy H relative to A between the identity id_X, and a retraction r. That is, $H(x,0) = x$, $H(x,1) \in A$, and $H(a,1) = a$ for every $x \in X$ and $a \in A$. A deformation retract captures the idea of continuously compressing a space onto a subspace: $H(-,0)$ is the identity on X and as the time parameter increases to 1, this mapping continuously shrinks down to the identity on A. Note that every deformation retract induces a retract $H(-,1) : X \to X$, but in general, retracts need not be deformation retracts – for instance, the constant map in any non-empty space is a retract, but is a deformation retract only if the space is contractible. Note also that a deformation retract induces a homotopy equivalence $A \simeq X$.

1.5 Quotient Spaces

Given a topological space X, we can endow a topology on any subset $A \subseteq X$ called the *subspace topology* by taking the open sets of A to be the open sets of X intersected with U.

This construction has the following universal property: if Y is any topological space and $f : Y \to A$ is a function, then f is continuous if and only if the composite

$$Y \xrightarrow{\quad f \quad} A \xhookrightarrow{\quad \iota \quad} X$$

is continuous as a function $Y \to X$, where ι is the canonical inclusion of A into X.

A similar construction can be performed for quotients; that is, a surjective set map $\pi : X \to B$. A topology can be placed on B by declaring that a set $U \subseteq B$ is open if and only if $\pi^{-1}[U]$ is open in X. This construction is called the *quotient topology* on B, and has the following universal property dual to that of subspaces: if Y is any topological spaec and $g : B \to Y$ is a function, then g is continuous if and only if the composite

$$X \xrightarrow{\quad \pi \quad} B \xhookrightarrow{\quad g \quad} Y$$

is continuous as a function $X \to Y$.

We sometimes prefer to describe a quotient as an equivalence class on X. This characterisation is equivalent to a surjective map $\pi : X \to B$, as given such a π, we may define an equivalence relation $\sim$ by $x \sim y$ if and only if $\pi(x) = \pi(y)$, so elements with the same image are identified under this relation. Conversely, given $\sim$ on X, we define B to be the set of equivalence classes and π to be the map defined by $x \mapsto [x]$.

If we have $A \subseteq X$, then we can define an equivalence relation $\sim$ such that $x \sim y$ if and only if both $x,y \in A$ or $x = y$. That is, every point in A is identified under $\sim$, and every point in $X \setminus A$ lies within its own singleton equivalence class. By a small abuse of notation, the resulting quotient space is denoted by X/A. Intuitively, this is the quotient space obtained by contracting all of A into a single point.

1.6 Gluing and CW Complexes

Given spaces X and Y, a subspace $A \subseteq X$, and a map $f : X \to Y$, we can form the space

$$X \cup_f Y := X \sqcup Y/\sim$$

where $\sim$ is the equivalence relation defined by $x \sim f(x)$ for all $x \in A$. This space is equipped with the quotient topology via the surjective map $X \sqcup Y \to X \cup_f Y$, where $X \sqcup Y$ has the disjoint union topology.

We will mostly be studying a important class of spaces built from this *gluing* process called *CW complexes* (where C stands for closure-finite and W for weak topology). Informally, these are spaces constructed by recursively gluing together discs of various dimensions.

Formally, we begin with a 0-*skeleton* consisting of a disjoint union $X^0 = \bigsqcup_i D_i^0$ of 0-discs, or points. Then given an $(n-1)$-skeleton X^{n-1}, we glue a collection of n-discs $\{D_j^n\}$ via *attaching maps* $\varphi_j : \partial D_j^n = S_j^{n-1} \to X^{n-1}$.

That is, given the maps φ_j, we define the n-skeleton X^n to be the space

$$X^n = X^{n-1} \bigcup_{\bigsqcup_j \varphi_j} \bigsqcup_j D_j^n$$

The attaching maps of each D_j^n canonically extend to maps $\varphi : D_j^n \to X^n$, and the images of these maps are called the *n-cells* of X, and the extension of φ_j is called the *characteristic map* of this n-cell.

This recursion then either stops at some finite level n, yielding a CW complex $X := X^n$, or continuing infinitely with arbitrarily high dimensional discs, in which case we define $X := \bigcup_n X^n$, with a subset $U \subseteq X$ being open if and only if $U \cap X_n$ is open for all n.

1.7 Group Theory

1.7.1 Free Products

Let $\{G_\alpha\}_\alpha$ be a collection of groups. A *word* on these groups is a finite sequence $g_1 \cdots g_m$ of elements $g_i \in G_{\alpha_i}$, and m is the *length* of the word. The empty word of length 0 is denoted by ε. The *product* of two words is their concatenation

$$(g_1 \cdots g_m) * (h_1 \cdots h_n) = g_1 \cdots g_m h_1 \cdots h_n$$

A word is *reduced* if it does not contain the identity of any group, and if every pair of consecutive letters are not from the same group.

Given any word g on the groups $\{G_\alpha\}_\alpha$, we can *reduce* it to a reduced word g' by removing all identity elements and replacing any consecutive elements g_i,g_{i+1} from the same group with their group product $g_i \cdot g_{i+1}$.

Let $*_\alpha G_\alpha$ be the set of reduced words on $\{G_\alpha\}_\alpha$. We can define an operation on this set as follows: given reduced words $g = g_1 \cdots g_m$ and $h = h_1 \dots h_n$, construct a new reduced word $g \bullet h$ by taking the concatenation $g * h$, then reduce the word recursively

$$g \bullet h = \begin{cases} gh & g_m \in G_\alpha, h_1 \in G_\beta,\, G_\alpha \neq G_\beta \\ g_1 \cdots g_{m-1}(g_m \cdot h_1)h_2 \cdots h_n & g_m, h_1 \in G_\alpha,\, g_m \cdot h_1 \neq \mathrm{id}_{G_\alpha} \\ g_1 \cdots g_{m-1} \bullet h_1 \cdots h_n & g_m, h_1 \in G_\alpha,\, g_m \cdot h_1 = \mathrm{id}_{G_\alpha} \end{cases}$$

Then, $(*_\alpha G_\alpha, \bullet)$ is a group called the *free product* of $\{G_\alpha\}_\alpha$, with identity ε, and the inverse of an element $g_1 \cdots g_m$ is given by $g_m^{-1} \cdots g_1^{-1}$.

1.7.2 Cokernels

Given a group homomorphism $\phi : A \to B$ of abelian groups, we have two fundamental subgroups, given by the image $\operatorname{im}(\phi) \leq B$, and the kernel $\ker(T) \trianglelefteq A$. A third fundamental subspace is given by the *cokernel*, defined as the quotient

$$\operatorname{coker}(\phi) \coloneqq B/\operatorname{im}(\phi)$$

For intuition on this definition, note that this definition makes sense for linear maps and vector spaces. A linear map $T : A \to B$ is a way to transform A into B. The kernel can be viewed as the space of elements in A that are "destroyed" by T. Then, the cokernel can be viewed as the space of elements in B that are "created" by T, in the sense that A is mapped to $\operatorname{im}(A) \subseteq B$, so any other element in B is new.

1.7.3 Smith Normal Form and the Structure Theorem for Finitely Generated Abelian Groups

Given two finite-dimensional vector spaces V and W over a field K and any linear map $T : V \to W$, there exist bases of V and W with respect to which the matrix of T is a block matrix of the form

$$\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$$

where r is the rank of T. A slightly weaker result holds if instead of vector spaces, we work with finitely generated free modules over $\mathbb{Z}$. Note that these modules are exactly the finitely generated abelian groups, so we phrase this theorem in terms of groups.

Given any group homomorphism $\phi : \mathbb{Z}^n \to \mathbb{Z}^m$, there exist bases of $\mathbb{Z}^n$ and $\mathbb{Z}^m$ and a diagonal matrix of the form

$$D = \begin{bmatrix} d_1 & & & \\ & d_2 & & \\ & & \ddots & \\ & & & d_r \end{bmatrix}$$

with *invariants* $d_i \in \mathbb{Z}^+$ and $d_1 \mid d_2 \mid d_3 \mid \ldots \mid d_r$ (where $\mid$ is the divides relation), such that the matrix of ϕ with respect to these bases is a block matrix of the form

$$\Sigma = \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}$$

That is, for any $m \times n$ matrix M with entries in $\mathbb{Z}$, there exist change of basis matrices $P \in \mathbb{Z}^{m\times m}$ and $Q \in \mathbb{Z}^{n\times n}$ such that $PMQ = \Sigma$ is of the above form, called the Smith normal form of M.

This form is convenient for calculating the kernel and cokernel of φ. First note that $r \leq \min(n,m)$. Then,

$$\ker\varphi \cong \mathbb{Z}^{n-r}$$

(note, $n - r$ is the number of zero columns) and

$$\operatorname{coker}\varphi \cong \left(\bigoplus_{i=1}^{r} \mathbb{Z}/d_r\right) \oplus \mathbb{Z}^{m-r}$$

(note, $m - r$ is the number of zero rows).

Since every finitely generated abelian group is the cokernel of some map, we see that every finitely generated abelian group must be of this form:

$$A \cong \left(\bigoplus_{i=1}^{r} \mathbb{Z}/d_r\right) \oplus \mathbb{Z}^{k}$$

and we call k the *rank* of A, written as $k = \operatorname{rk}_{\mathbb{Z}}(A)$.

2 Introduction

The goal of algebraic topology is to translate questions in topology into questions in algebra, most commonly by constructing algebraic invariants of topological spaces. That is, given a space X, we wish to construct an algebraic structure $A(X)$ such that if spaces X and Y are homeomorphic (or just homotopy equivalent), then the associated algebraic objects are isomorphic:

$$X \simeq Y \quad \longrightarrow \quad A(X) \cong A(Y)$$

For an algebraic invariant $A(-)$ to be useful, we require that:

1. it is "easy" to compute $A(-)$ and to tell when the algebraic objects are not isomorphic;
2. the algebraic invariant is "fine" enough in that $A(X) \ncong A(Y)$ often, for non-homeomorphic X and Y.

We have previously constructed the fundamental group (or first homotopy group) $\pi_1(-)$. This invariant takes a pointed space (X,x) as input and returns the group of homotopy classes of pointed maps $(S^1,*) \to (X,x)$, or loops in X based at x, under the operation of path concatenation.

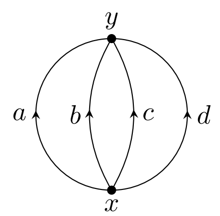
The fundamental group is a complete invariant for compact surfaces, but fails to capture the topology of higher-dimensions, only detecting the 1-dimensional hole structure of a space. For instance, the fundamental groups of $\mathbb{R}^3$ and $\mathbb{R}^4$ are both trivial and cannot be used to distinguish them. Fundamental groups are also in general non-abelian, so it is difficult to determine whether two such groups are non-isomorphic.

One natural generalisation of this is to consider pointed maps not just from the circle S^1, but from n-spheres into a pointed map: given a pointed space (X,x), the *nth homotopy group* $\pi_n(X,x)$ is the group of homotopy classes of pointed maps $(S^n,*) \to (X,x)$. The nth homotopy groups are a much more fine invariant, and can tell apart many topological spaces, but they are extremely difficult to compute. Even for simple spaces like spheres, the homotopy groups are generally unknown.

Here, we study a more computable alternative: *homology groups*. These invariants are abelian groups and are easier to distinguish and compute than homotopy groups – for instance, the homology groups of spheres are all known – but conversely, they contain less information than homotopy groups.

2.1 Homology

Consider the following graph, X_1, consisting of two 0-cells connected with four oriented 1-cells:



The fundamental group of X_1 consists of loops formed by sequences of edges, starting and ending at some fixed basepoint. For instance, at the basepoint x, one possible loop is given by ab^{-1}, travelling along a, then along b in reverse direction. Another loop is given by $ad^{-1}bc^{-1}ac^{-1}$. Because these loops must be continuous paths, the fundamental group is generally non-abelian.

To simplify, let us consider what happens if we abelianise this group. For example, the loops ab^{-1} and $b^{-1}a$ are equal if we allow a to commute with b^{-1}. Note that these loops are really the same circle, just with a different basepoint – x for ab^{-1} and y for $b^{-1}a$. Choosing a new basepoint in a loop just cyclically permutes its edges, so we no longer have to consider pointed spaces: instead of loops, we have *cycles*, independent of a choice of basepoint.

Now working with an abelian group, we swap to additive notation, so cycles are $\mathbb{Z}$-linear combinations of edges. More generally, a (1-)*chain* is any such linear combination of edges. Even more generally, for any cell complex X, an *n-chain* in X is a linear combination of n-cells – that is, the group of n-chains in a space X with k n-cells $\{c_i\}_{i=1}^k$ is the free abelian group $C_n(X) = \bigoplus_{i=1}^k \mathbb{Z} \cdot c_i \cong \mathbb{Z}^k$ on the basis $\{c_i\}$.

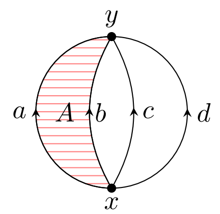
Note that not all 1-chains may be interpreted as paths, as endpoints do not have to match up. For instance $a + b$ is a chain, but not a meaningful path. In any case, the order of concatenation of edges is immaterial in an 1-chain. Note also that chains may have multiple decompositions into cycles: for instance, $(a - b) + (c - d) = (a - d) + (c - b)$, so more generally, we define a *cycle* to be any chain that has at least one decomposition into a cycle of the previous geometric sense. How can we determine when a chain admits such a decomposition?

In a geometric cycle, interpreted as a path, every vertex is entered and exited the same number of times. In the above graph, given a chain $\alpha a + \beta b + \gamma c + \delta d$, the net number of times y is entered is $\alpha + \beta + \gamma + \delta$, and similarly, the net number of times x is entered is $-\alpha - \beta - \gamma - \delta$. For a chain to be a cycle, we require that these quantities are simultaneously zero, so in the above graph, a chain is a cycle if and only if $\alpha + \beta + \gamma + \delta = 0$.

Let C_1 be the free abelian group with basis a,b,c,d, and C_0 the free abelian group with basis x,y. Elements of C_1 are then linear combinations of edges, which are exactly the 1-chains, and similarly, elements of C_0 are linear combinations of vertices, or *0-chains*.

We define the *(1st) boundary* homomorphism $\partial_1 : C_1 \to C_0$ by sending each basis element (edge) to the vertex at its head, minus its vertex at the tail. For instance, for the graph above, every edge is sent to $y - x$, as every edge points from x to y. Then, the action of this homomorphism on a chain $\alpha a + \beta b + \gamma c + \delta d$ is given by $(\alpha + \beta + \gamma + \delta)y - (\alpha + \beta + \gamma + \delta)x$. Thus, the cycles are precisely the kernel of ∂_1. It is easy to verify that $a - b$, $b - c$, and $c - d$ form a basis for this kernel – so every cycle in X_1 is a linear combination of these three cycles. In this way, this kernel captures the geometric information that the graph X_1 has three "(1-dimensional) holes".

Let us expand the graph by attaching a 2-cell, A, along the cycle $a - b$ to produce a 2-dimensional cell complex X_2.



We similarly define the group C_2 to be the free abelian group with basis A. We can also define another boundary operator $\partial_2 : C_2 \to C_1$, but this requires a choice of orientation for A.

If we regard A as being oriented clockwise, its boundary is then the cycle $a - b$. This cycle now no longer encloses a hole as it did in X_1, as it can be linearly contracted to a point over A. This suggests that we form a quotient of the group of cycles in the previous example by factoring out the subgroup generated by $a - b$. For instance, the cycles $a - c$ and $b - c$ would now be equivalent in this quotient, consistent with them being homotopic in X_2. This quotient group is exactly $\ker \partial_1 / \operatorname{im} \partial_2$ – the 1-cycles modulo those that are boundaries of 2-cells. This quotient group is the *homology group* $H_1(X_2)$. In this case, $H_1(X_2)$ is free abelian on 2 generators, corresponding to filling A having removed one of the three holes.

We can also compute $H_1(X_1)$ by taking $C_2 = 0$ to be the trivial group, as there are no 2-cells in X_1, and ∂_2 to be the trivial homomorphism; so we have $H_1(X_1) = \ker \partial_1 / \operatorname{im} \partial_2 = \ker \partial_1$ is free abelian on 3 generators, corresponding to our three 1-dimensional holes.

We could attach another 2-cell along the same cycle $a - b$, forming a kind of hollow banana shape in X_3. $H_1(X_3)$ is unchanged, but now ∂_2 has a non-trivial kernel – the group generated by the spherical 2-cycle $A - B$. Just as the three cycles in X_1 detected 1-dimensional holes, the presence of the 2-cycle $A - B$ indicates the existence of a 2-dimensional hole – the missing interior of this sphere. We could expand this cell complex again, attaching a 3-cell C along $A - B$. This creates a new chain group C_3, and we can define a boundary homomorphism $\partial_3 : C_3 \to C_2$ by sending C to $A - B$ (note that this again depends on a choice of orientation for C). Now, $H_2(X_4) = \ker \partial_2 / \operatorname{im} \partial_3$ is trivial, as this 2-hole has now been filled in.

3 Simplicial Homology

The general pattern is now clear: for any cell complex, we have chain groups $C_n(X)$ of n-chains in X, and boundary operators $\partial_n : C_n(X) \to C_{n-1}(X)$, from which we define the nth homology group $H_n(X) = \ker \partial_n / \operatorname{im} \partial_{n+1}$.

The difficulty is in how we define ∂_n in general. For $n = 1$, this is not hard: the boundary of an edge is the vertex at its head minus the vertex at its tail. However, for arbitrary n, this becomes rather complicated.

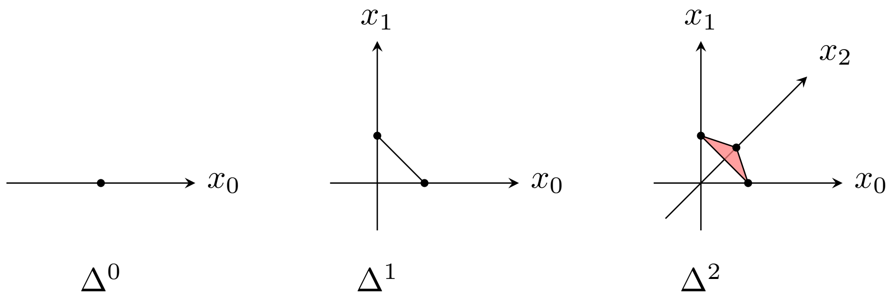
For now, we start with a simplified version of homology called *simplicial homology*, where we deal only with topological spaces that can be expressed in terms of *simplices* which admit an easier notion of boundary mapping.

3.1 Δ-Complexes

The *standard n-simplex* $\Delta^n \subseteq \mathbb{R}^{n+1}$ is the subspace

$$\Delta^n := \left\{ \mathbf{x} \in \mathbb{R}^{n+1} : x_i \geq 0, \sum_{i=0}^{n} x_i = 1 \right\}$$

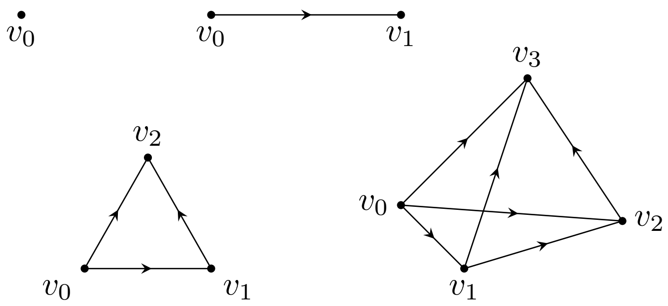
whose vertices $v_0, v_1, \ldots, v_n$ are the unit vectors along the coordinate axes. More generally, any homeomorphic space will also be called an n-simplex and labelled Δ^n.



The standard n-simplex for $n = 0,1,2$

From now, on we suppress the coordinate axes and draw general (non-standard) simplices (particularly as drawing axes in > 3 dimensions is rather difficult!).

For the purposes of simplicial homology, it is important to keep track of the ordering of these vertices, so we also refer to a simplex by an ordered list of its vertices: $[v_0, \ldots, v_n]$. This representation also has a side effect of determining the orientation of its edges $[v_i, v_j]$ according to increasing subscripts. When drawing simplices, the convention is to annotate the edges with an arrow pointing in ascending order of vertices:



Deleting one of the $n+1$ vertices of an n-simplex $\Delta^n = [v_0, \ldots, v_n]$, say v_j, the remaining n vertices span an $(n-1)$-simplex $[v_0, \ldots, \widehat{v_j}, \ldots, v_n]$ (where $\widehat{\ }$ indicates omission) called the jth *face* of $[v_0, \ldots, v_n]$, denoted by $\partial_j \Delta^n$.

More concretely, in terms of the standard n-simplex, the jth face of Δ^n is the subspace $\partial_j \Delta^n \subseteq \Delta^n$ of points whose jth coordinate is zero. That is,

$$\begin{aligned}\partial_j \Delta^n &= \{\mathbf{x} \in \Delta^n : x_j = 0\} \\ &= \left\{\mathbf{x} \in \mathbb{R}^{n+1} : x_j = 0, x_i \geq 0 \sum_{i=0}^{n} x_i = 1, \right\}\end{aligned}$$

Note that, geometrically, the jth face is the one *opposite* the deleted jth vertex.

For instance, in the above simplices: the 0th face of Δ^1 is the vertex v_1, and the 1st face is v_0; the 0th face of Δ^2 is the edge $[v_1, v_2]$, the 1st face the edge $[v_0, v_2]$, and the 2nd face the edge $[v_0, v_1]$; and the 0th face of Δ^3 is the triangle $[v_1, v_2, v_3]$, etc.

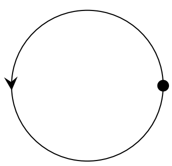
The *boundary* $\partial \Delta^n$ of Δ^n is then the union of its $n+1$ faces. Note that the unique face $\partial_0 \Delta^0$ of Δ^0 is the empty set, so $\partial \Delta^0 = \emptyset$.

A Δ-*complex* X is a topological space* defined inductively as follows:

1. Start with a collection of 0-simplices, or points. This is the 0-skeleton X^0.
2. Inductively, the n-skeleton X^n is obtained from X^{n-1} by attaching n-simplices Δ^n_α where each face $\partial_i \Delta^n_\alpha$ is identified with an $(n-1)$-simplex Δ^{n-1}_β in X^{n-1}.
3. If k is the minimal k such that $X^k = X^{k+1}$, i.e., there are no m-cells added for any $m > k$, then $X = X^k$ has dimension k.

   More generally, $X = \bigcup_{n \in \mathbb{N}} X^n$, in which case, a subspace $U \subseteq X$ is open if and only if $U \cap X^n \subseteq X^n$ is open for all n.

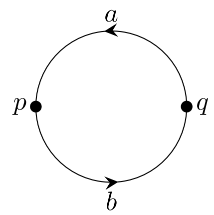
Example. Start with a single point $X^0 = \Delta^0$, and attach a single 1-simplex Δ^1 in the only possible way. That is, both boundary points are identified with the 0-skeleton:



△

Example. Now start with two points $X^0 = \{p, q\}$, and attach two 1-simplices a, b as follows:

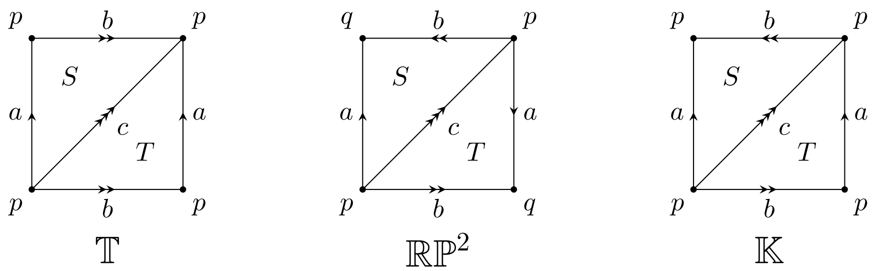
*More properly, a Δ-complex *structure on* a space X.



with the arrows indicating that the 0th face of a is identified with p, the 1st face with q; and the 0th face of b is identified with q, and the 1st face with p.

Note that both of these spaces are homeomorphic to a circle, so a topological space can have multiple Δ-complex structures. △

Example. The torus $\mathbb{T}$, real projective plane $\mathbb{RP}^2$, and the Klein bottle $\mathbb{K}$ can all be constructed as quotients of a square by identifying opposite edges. They can all be constructed as Δ-complexes as follows:



△

A Δ-complex is essentially just combinatorial data. That is, it is determined up to homeomorphism by the sets of n-simplices, S_n, $n \geq 0$, together with the attaching rules – namely, the *face maps* $d_i^n : S_n \to S_{n-1}$, $0 \leq i \leq n$, specifying that $\partial_i \Delta_\alpha^n$ is identified with $\Delta^{n-1}_{d_i^n(\alpha)}$. These maps are not arbitrary, but satisfy the relation

$$d_i^{n-1} \circ d_j^n = d_{j-1}^{n-1} \circ d_i^n$$

whenever $i < j$.

Writing the simplex as $[v_0, \ldots, v_i, \ldots, v_j, \ldots, v_n]$, this relation is just saying that removing v_j, then v_i, should be the same as removing v_i, then v_j; removing v_i first means that v_j is the $(j-1)$th vertex in the intermediary simplex.

Such a collection of combinatorial data $S = (S_\bullet, d_\bullet^\bullet)$ is called a Δ-*set* or *semi-simplicial set.*

Given a Δ-set S, we denote the associated Δ-complex, called its *geometric realisation*, by $|S|$. More precisely, a Δ-complex is really a topological space X equipped with a homeomorphism $X \cong |S|$ for some Δ-set S, the latter of which is then called a Δ-complex structure on X. As in the earlier examples of circles, a topological space can admit distinct Δ-complex structures.

Example. In the torus $\mathbb{T}$ above, we have

$$S_0 = \{p\}, \qquad S_1 = \{a,b,c\}, \qquad S_2\{S,T\}$$

with face maps

$$\begin{aligned} d_0^2(S) &= b, & d_0^2(T) &= a, \\ d_1^2(S) &= c, & d_1^2(T) &= c, \\ d_2^2(S) &= a, & d_2^2(T) &= b; \end{aligned} \qquad \begin{aligned} d_0^1(a) = d_0^1(b) = d_0^1(c) = p, \\ d_1^1(a) = d_1^1(b) = d_1^1(c) = p; \end{aligned} \qquad d_0^0(p) = \emptyset$$

△

3.2 Simplicial Homology

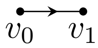
Previously, we defined some homology groups on simple CW complexes in terms of certain *boundary operators*. For arbitrary CW complexes, defining these operators for higher dimensions is tricky, but for Δ-complexes, the situation is a little easier.

- For a 0-simplex, i.e. a point,

  the boundary is empty.

- For a 1-simplex,

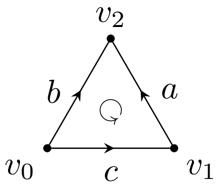
  

  just like we did for CW complexes, we define its oriented to be the formal difference between the vertices at its head and its tail.

  $$v_1 - v_0$$

  Note that the choice of orientation is arbitrary, and $v_0 - v_1$ would work just as well.

- For a 2-simplex,

  we define its oriented boundary as

  $$a - b + c$$

  Again, this choice is arbitrary, and orienting clockwise works equally well.

- Similarly, for a general n-simplex s, the boundary is then the alternating sum of its faces:

  $$\partial_n(s) = \sum_{i=0}^{n}(-1)^i d_i^n(s)$$
  $$\partial_n\big([v_0,\dots,v_n]\big) = \sum_{i=0}^{n}(-1)^i[v_0,\dots,\widehat{v_i},\dots,v_n]$$

  where $\widehat{\ }$ indicates omission.

From this point, the theory is entirely the same as in the introduction:

Let S be a Δ-set.

- The group of *n-chains* in S is the free abelian group on S_n, denoted by $\Delta_n(S)$.
- The *boundary operator* $\partial : \Delta_n(S) \to \Delta_{n-1}(S)$ is the homomorphism given on the generators $s \in S_n$ by the formula above, noting that $\Delta_{-1}(S)$ is the trivial group 0, and that ∂_0 is the zero map.
- The group of *n-cycles* $Z_n(S)$ is the kernel of the nth boundary operator,

  $$Z_n(S) := \ker(\partial_n)$$

- The group of *n-boundaries* $B_n(S)$ is the image of the $(n+1)$th boundary operator,

$$B_n(S) \coloneqq \operatorname{im}(\partial_{n+1})$$

- The *nth simplicial homology group* $H_n^\Delta(S)$ is the group of n-cycles modulo those that are boundaries,

$$H_n^\Delta(S) \coloneqq \frac{Z_n(S)}{B_n(S)} = \frac{\ker(\partial_n)}{\operatorname{im}(\partial_{n+1})}$$

  and elements of this group are called *homology classes*.

Of course, for this last expression to hold, we require that $B_n(S) \subseteq Z_n(S)$ for all n and all Δ-sets S.

Lemma 3.1. *Let S be a Δ-set. Then, $\partial_n \circ \partial_{n+1} = 0$. Equivalently, $B_n(S) \subseteq Z_n(S)$.*

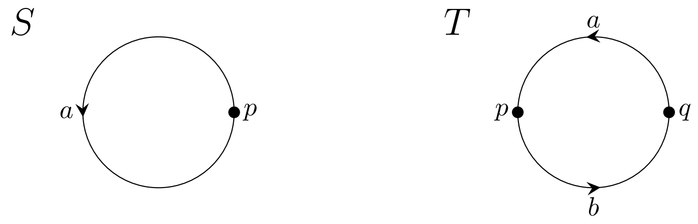
Proof. The equivalence of the two statements is clear from the definition of cycles and boundaries.

Let $s \in S_{n+1}$. Then,

$$\begin{aligned}
\partial_n \partial_{n+1}(s) &= \partial_n \left(\sum_{j=0}^{n+1} (-1)^j d_j^{n+1}(s) \right) \\
&= \sum_{i=0}^{n} \sum_{j=0}^{n+1} (-1)^{i+j} d_i^n d_j^{n+1}(s) \\
&= \sum_{0 \le i < j \le n+1} (-1)^{i+j} d_i^n d_j^{n+1}(s) \qquad + \qquad \sum_{0 \le j \le i \le n} (-1)^{i+j} d_i^n d_j^{n+1}(s) \\
&= \sum_{0 \le i < j \le n+1} (-1)^{i+j} d_{j-1}^n d_i^{n+1}(s) \qquad + \qquad \sum_{0 \le j \le i \le n} (-1)^{i+j} d_i^n d_j^{n+1}(s) \\
&= \sum_{0 \le i < j \le n+1} (-1)^{i+j-1} d_j^n d_i^{n+1}(s) \qquad + \qquad \sum_{0 \le j \le i \le n} (-1)^{i+j} d_i^n d_j^{n+1}(s) \\
&= 0
\end{aligned}$$

■

Example. We calculate the homology groups of the following Δ-sets.



For S, we have boundary operators

$$\partial_1 : \mathbb{Z}a = \Delta_1(S) \to \Delta_0(S) = \mathbb{Z}p$$
$$\partial_0 : \mathbb{Z}p = \Delta_0(S) \to \Delta_{-1}(S) = 0$$

respectively defined on the generators a and p by

$$\partial_1(a) = p - p = 0$$
$$\partial_0(p) = 0$$

with all other boundary operators trivially the zero map, as there are no simplices of any other dimension. These boundary operators both vanish, so $Z_1(S) = Z_0(S) = \mathbb{Z}$, and $B_n(S) = 0$ for all n. So, $H_0(S) = H_1(S) = \mathbb{Z}/0 \cong \mathbb{Z}$.
For T, we have boundary operators

$$\partial_1 : \mathbb{Z}a \oplus \mathbb{Z}b = \Delta_1(S) \to \Delta_0(S) = \mathbb{Z}p \oplus \mathbb{Z}p$$
$$\partial_0 : \mathbb{Z}p \oplus \mathbb{Z}q = \Delta_0(S) \to \Delta_{-1}(S) = 0$$

defined by

$$\partial_1(a) = p - q, \qquad \partial_0(p) = 0,$$
$$\partial_1(b) = q - p; \qquad \partial_0(q) = 0$$

which we can represent more compactly by

$$\cdots \longrightarrow 0 \longrightarrow \mathbb{Z}a \oplus \mathbb{Z}b \xrightarrow[\begin{array}{c|cc} & a & b \\ \hline p & 1 & -1 \\ q & -1 & 1 \end{array}]{\partial_1} \mathbb{Z}p \oplus \mathbb{Z}q \xrightarrow[\begin{array}{c|cc} & p & q \\ \hline * & 0 & 0 \end{array}]{\partial_0} 0$$

with all other n-chains trivial. By inspection, we find that $\ker(\partial_1) = \mathbb{Z}(a+b)$ and $\operatorname{im}(\partial_1) = \mathbb{Z}(p-q)$, but we can do this more generally by examining the Smith normal form of the matrix associated with ∂_1:

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

Along the diagonal, we have a single 1, so $\operatorname{im}(\partial_1) \cong \mathbb{Z}$, and the remaining zero row gives $\ker(\partial_1) \cong \mathbb{Z}^{2-1} = \mathbb{Z}$. The zero matrix for ∂_0 also gives $\ker(\partial_0) = \mathbb{Z}p \oplus \mathbb{Z}q \cong \mathbb{Z}^2$.

$$H_0(T) = \frac{\ker(\partial_0)}{\operatorname{im}(\partial_1)} = \frac{\mathbb{Z}^2}{\mathbb{Z}} = \mathbb{Z} \qquad H_1(T) = \frac{\ker(\partial_1)}{\operatorname{im}(\partial_2)} = \frac{\mathbb{Z}}{0} = \mathbb{Z}$$

and all other homology groups 0. △

We just saw that these two Δ-sets S and T, which have the same geometric realisation, have the same homology groups. This is not a coincidence. Though it is not clear at all at this point, it turns out that homology is an invariant of the geometric realisation.

If X is a topological space with a Δ-complex structure $|S| \cong X$, we define its *nth simplicial homology group* to be

$$H_n^{\Delta}(X) \coloneqq H_n(S)$$

Note that, once we have the chain groups and boundary operators, computing the simplicial homology is an entirely mechanical process:

Algorithm 1 Simplicial Homology

1: Determine the matrix of each boundary operator

$$\partial_n\big([v_0, \dots, v_n]\big) = \sum_{i=0}^{n} (-1)^i [v_0, \dots, \widehat{v_i}, \dots, v_n]$$

2: Determine the Smith normal form for each boundary operator.

3: For each pair of matrices

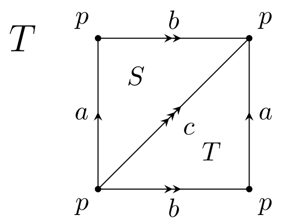
$$\mathbb{Z}^{\ell} \xrightarrow{\mathbf{A}} \mathbb{Z}^m \xrightarrow{\mathbf{B}} \mathbb{Z}^n$$

with $\mathbf{BA} = 0$, we have

$$\frac{\ker \mathbf{B}}{\operatorname{im} \mathbf{A}} \cong \left(\bigoplus_{i=1}^{r} \mathbb{Z}/d_i \right) \oplus \mathbb{Z}^{m-a-b}$$

where $\{d_i\}_{i=1}^{r}$ are the invariants of $\mathbf{A}$ (i.e. the diagonal elements), and $a = \operatorname{rank}(\mathbf{A})$ and $b = \operatorname{rank}(\mathbf{B})$ (i.e. the number of invariants of $\mathbf{A}$ and $\mathbf{B}$, respectively).

Example. We compute the simplicial homology of the torus



We have the chain of boundary operators

$$\cdots \longrightarrow 0 \longrightarrow \mathbb{Z}S \oplus \mathbb{Z}T \xrightarrow{\qquad \partial_2 \qquad} \mathbb{Z}a \oplus \mathbb{Z}b \oplus \mathbb{Z}c \xrightarrow{\qquad \partial_1 \qquad} \mathbb{Z}p \xrightarrow{\partial_0} 0$$

$$\begin{array}{c c} & \begin{array}{cc} S & T \end{array} \\ \begin{array}{c} a \\ b \\ c \end{array} & \begin{bmatrix} -1 & 1 \\ -1 & 1 \\ 1 & -1 \end{bmatrix} \end{array} \qquad\qquad \begin{array}{c c} & \begin{array}{ccc} a & b & c \end{array} \\ p & \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} \end{array}$$

where ∂_2 has Smith normal form

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

and ∂_1 already in Smith nornal form, so we have $\operatorname{rank}(A) = 1$ and $\operatorname{rank}(B) = 0$, giving,

$$\begin{aligned} H_1(T) &= \left(\bigoplus_{i=1}^{1} \mathbb{Z}/1 \right) \oplus \mathbb{Z}^{3-1-0} \\ &= \mathbb{Z}/1 \oplus \mathbb{Z}^2 \\ &= \mathbb{Z}^2 \end{aligned}$$

For $H_0(T)$, ∂_1 has no invariants; ∂_1 has rank 0; and ∂_0 is the zero map which also has rank 0, so

$$\begin{aligned} H_0(T) &= \mathbb{Z}^{1-0-0} \\ &= \mathbb{Z} \end{aligned}$$

and for $H_2(T)$, ∂_3 is the zero map with no invariants, so

$$\begin{aligned} H_2(T) &= \mathbb{Z}^{1-0-0} \\ &= \mathbb{Z} \end{aligned}$$

All other boundary maps are trivial, so all other homology groups are 0, giving

$$H_n^{\Delta}(T) \cong \begin{cases} \mathbb{Z} \oplus \mathbb{Z} & n = 1 \\ \mathbb{Z} & n = 0,2 \\ 0 & n \geq 3 \end{cases}$$

$\triangle$

3.3 Chain Complexes

So far, we have been computing homology groups of Δ-sets, first by mapping them to free abelian groups of n-chains with boundary operators between them, before finally computing the homology groups in terms of these operators.

$$S \quad \mapsto \quad (\Delta_\bullet(S), \partial_\bullet) \quad \mapsto \quad H_\bullet(S)$$

Algebraically, this middle step takes the form of sequences of abelian groups with homomorphisms between them,

$$\cdots \to \Delta_{n+1}(S) \xrightarrow{\partial_{n+1}} \Delta_n(S) \xrightarrow{\partial_n} \Delta_{n-1}(S) \to \cdots \to \Delta_1(S) \xrightarrow{\partial_1} \Delta_0(S) \xrightarrow{\partial_0} 0$$

It will be useful to discuss this structure independently from the specific situation here, as this process will recur repeatedly in the future in multiple different contexts.

A *chain complex* $C_\bullet = (C_\bullet, \partial_\bullet)$ is a family of abelian groups $(C_n)_{n \in \mathbb{Z}}$ equipped with maps called *differentials* $\partial_n : C_n \to C_{n-1}$,

$$\cdots \to C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \to \cdots$$

such that $\partial_n \circ \partial_{n+1} = 0$ for each n.

In general, if we only define C_n for $n \in [a,b]$, then it's understood that $C_n = 0$ for all $n \notin [a,b]$, and we call such a chain complex *bounded* (or *bounded above/below*, if instead defined on a half-infinite interval).

Example. Let S be a Δ-set. Then, the collection of chain groups in S and the boundary operators between them form a (bounded below) chain complex $\Delta_\bullet(S)$ called the *simplicial chain complex* associated with S. $\triangle$

Let $C_\bullet$ be a chain complex.

- The *n-cycles* are
$$Z_n(C_\bullet) = \ker(\partial_n)$$
- The *n-boundaries* are
$$B_n(C_\bullet) = \operatorname{im}(\partial_{n+1})$$

and since $\partial_n \circ \partial_{n+1} = 0$, we have $B_n \subseteq Z_n$, so

- The *nth homology group* is
$$H_n(C_\bullet) \coloneqq \frac{Z_n}{B_n} = \frac{\ker(\partial_n)}{\operatorname{im}(\partial_{n+1})}$$
- If $H_n(C_\bullet) = 0$, or equivalently, if $\operatorname{im}(\partial_{n+1}) = \ker(\partial_n)$, then we say that $C_\bullet$ is *exact in degree n*. If $C_\bullet$ is exact in all degrees, then we say that $C_\bullet$ is *exact*.

Let $(C_\bullet,\partial_\bullet)$ and $(D_\bullet,\partial'_\bullet)$ be chain complexes. A *chain map* $f_\bullet : C_\bullet \to D_\bullet$ is a family of maps $f_n : C_n \to D_n$, such that

$$\begin{tikzcd}
\cdots \arrow[r] & C_{n+1} \arrow[r, "\partial_{n+1}"] \arrow[d, "f_{n+1}"] & C_n \arrow[r, "\partial_n"] \arrow[d, "f_n"] & C_{n-1} \arrow[r] \arrow[d, "f_{n-1}"] & \cdots \\
\cdots \arrow[r] & D_{n+1} \arrow[r, "\partial'_{n+1}"] & D_n \arrow[r, "\partial'_n"] & D_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

commutes for all n. That is,
$$\partial'_n \circ f_n = f_{n-1} \circ \partial_n$$

Lemma 3.2. *A chain map $f_\bullet : C_\bullet \to C'_\bullet$ restricts to maps*

- *$f_n : Z_n(C_\bullet) \to Z_n(C'_\bullet)$;*
- *$f_n : B_n(C_\bullet) \to B_n(C'_\bullet)$,*

and hence induces maps $f_n : H_n(C_\bullet) \to H_n(C'_\bullet)$.

Proof. If $\partial_n(\alpha) = 0$, then $\partial'_n\big(f_n(\alpha)\big) = f_{n-1}\big(\partial_n(\alpha)\big) = 0$, so f_n sends cycles to cycles. Also, f_n sends boundaries to boundaries as $f_{n-1}\big(\partial_n(\beta)\big) = \partial'_n\big(f_n(\beta)\big)$. Hence, f_n induces homomorphisms in homology. ■

Let $S = (S_\bullet,d_\bullet)$ and $T = (T_\bullet,d'_\bullet)$ be two Δ-sets. A *map of Δ-sets* $f_\bullet : S \to T$ is a family of maps $f_n : S_n \to T_n$ such that every square in

$$\begin{tikzcd}
\cdots \arrow[r, shift left=2] \arrow[r, shift left=1] \arrow[r] \arrow[r, shift right=1] \arrow[r, shift right=2] & C_{n+1} \arrow[r, shift left=1.5, "d^{n+1}_\bullet"] \arrow[r, shift left=0.5] \arrow[r, shift right=0.5] \arrow[r, shift right=1.5] \arrow[d, "f_{n+1}"] & C_n \arrow[r, shift left=1, "d^n_\bullet"] \arrow[r] \arrow[r, shift right=1] \arrow[d, "f_n"] & C_{n-1} \arrow[r, shift left=0.5] \arrow[r, shift right=0.5] \arrow[d, "f_{n-1}"] & \cdots \\
\cdots \arrow[r, shift left=2] \arrow[r, shift left=1] \arrow[r] \arrow[r, shift right=1] \arrow[r, shift right=2] & T_{n+1} \arrow[r, shift left=1.5, "d'^{n+1}_\bullet"] \arrow[r, shift left=0.5] \arrow[r, shift right=0.5] \arrow[r, shift right=1.5] & T_n \arrow[r, shift left=1, "d'^n_\bullet"] \arrow[r] \arrow[r, shift right=1] & T_{n-1} \arrow[r, shift left=0.5] \arrow[r, shift right=0.5] & \cdots
\end{tikzcd}$$

commutes. That is, for all $0 \leq i \leq n$,

$$f_n \circ d_i^n = d_i^{n+1} \circ f_{n+1}$$

whenever both sides of the equation are defined.

Lemma 3.3. *A map of Δ-sets $f_\bullet : S \to T$ induces a chain map $f_\bullet : \Delta_\bullet(S) \to \Delta_\bullet(T)$.*

Proof. Define $f_n : \Delta_n(S) \to \Delta_n(T)$ on generators $s \in S_n$ by

$$\begin{aligned} \mathbb{Z}S_n &\longrightarrow \mathbb{Z}T_n \\ s &\longmapsto f_n(s) \end{aligned}$$

Then,

$$\begin{aligned} \partial'_n \circ f_n(s) &= \partial'_n\big(f_n(s)\big) \\ &= \sum_{i=0}^{n}(-1)^i d'^n_i\big(f_n(s)\big) \\ &= \sum_{i=0}^{n}(-1)^i f_{n-1}\big(d_i(s)\big) \\ &= f_{n-1}\left(\sum_{i=0}^{n}(-1)^i d_i(s)\right) \\ &= f_{n-1} \circ \partial_n(s) \end{aligned}$$

■

Combining the previous two results, we have,

Corollary 3.3.1. *Every map of Δ-sets induces a map in simplicial homology.*

4 Singular Homology

In the previous section, we defined the simplicial homology for Δ-complexes. That is, spaces equipped with homeomorphisms to a Δ-set. There are two main problems with this result. Firstly, topological spaces often do not have an obvious Δ-complex structure – and some topological spaces admit no such structure at all. Secondly, even if a given space admits a Δ-complex structure, it may not be unique, and we haven't yet proven that simplicial homology is independent of choice of Δ-complex structure.

We now present an alternative theory of homology that avoids these difficulties, and will eventually allow us to prove the independence mentioned above.

Let X be a topological space and let $n \geq 0$. A *singular n-simplex* in X is a continuous map $\sigma : \Delta^n \to X$.

Example.

1. A singular 0-simplex is a function $\sigma : \Delta^0 \cong \{*\} \to X$. Such a function just picks out a point $x \in X$ and we sometimes identify the two.
2. A singular 1-simplex is a function $\sigma : \Delta^1 \cong [0,1] \to X$, which is just a path in X from $\sigma(0)$ to $\sigma(1)$.
3. If X is a Δ-complex, then any n-simplex in X can be viewed as the image of a simplex Δ^n under a function into X, i.e. a singular n-simplex in X.

△

Now, recall the definition of the oriented boundary of a simplex in a Δ-complex:

$$\partial_n\big([v_0, \ldots, v_n]\big) = \sum_{i=0}^{n}(-1)^i[v_0, \ldots, \widehat{v_i}, \ldots, v_n]$$

Because a singular simplex $\sigma : \Delta^n = [v_0, \ldots, v_n] \to X$ is a map, the ith "face" of this simplex is just the restriction of σ to the ith face of the standard simplex. So, the oriented boundary of $\sigma : \Delta^n =$

$[v_0,\ldots,v_n] \to X$ is

$$\partial_n(\sigma) = \sum_{i=0}^{n}(-1)^i \sigma\big|_{\partial_i \Delta^n}$$

where ∂_i is the boundary operator for standard simplices found earlier. That is,

$$= \sum_{i=0}^{n}(-1)^i \sigma\big|_{[v_0,\ldots,\widehat{v_i},\ldots,v_n]}$$

Note that these faces that σ is restricted to are themselves homeomorphic to the standard $(n-1)$-simplex, so we can view each restriction as a singular $(n-1)$-simplex themselves, and thus this expression becomes a formal linear combination of singular $(n-1)$-simplices in X, i.e., an element of C_{n-1}.

Example. Let $\sigma : \Delta^2 = [v_0,v_1,v_2] \to X$ be a singular simplex. Then, the boundary of σ is

$$\partial(\sigma) = \sigma\big|_{[v_1,v_2]} - \sigma\big|_{[v_0,v_2]} + \sigma\big|_{[v_0,v_1]}$$

△

Let X be a topological space and $n \geq 0$.

- The group of *singular n-chains* in X is the free abelian group on the singular n-simplices, denoted by $C_n(X) \coloneqq \mathbb{Z} \cdot \{\sigma : \Delta^n \to X\}$.
- The *boundary operator* $\partial : C_n(X) \to C_{n-1}(X)$ is the homomorphism given on the generators $\sigma \in C_n(X)$ by the alternating sum of faces as above.

Note that the groups $C_n(X)$ are usually infinite, and frequently uncountable, as there are many ways to map a standard simplex into a space.

The same proof as for Δ-sets then translates across to singular simplices:

Lemma 4.1. *Let X be a topological space. Then, $\partial_n \circ \partial_{n+1} : C_{n+1}(X) \to C_{n-1}(X)$ is the zero map.*

That is, $(C_\bullet(X),\partial_\bullet)$ forms a chain complex called the *singular chain complex* associated with X. The *singular homology groups* of X are then the homology groups of this chain complex, i.e.,

$$H_n(X) \coloneqq H_n\big(C_\bullet(X)\big)$$

Example. Let $X = \{*\}$ be a point. For each $n \geq 0$, there is a unique singular n-simplex given by the constant map $c_n : \Delta^n \to X$ at the unique point of X, so the singular chain complex is

$$\cdots \to \mathbb{Z} \xrightarrow{\partial_n} \mathbb{Z} \to \cdots \to \mathbb{Z} \xrightarrow{\partial_2} \mathbb{Z} \xrightarrow{\partial_1} \mathbb{Z} \xrightarrow{\partial_0} 0$$

Now,

$$\begin{aligned}
\partial_n(c_n) &= \sum_{i=0}^{n}(-1)^i c_n|_{\partial_i \Delta^n} \\
&= \sum_{i=0}^{n}(-1)^n c_{n-1} \\
&= \begin{cases} c_{n-1} & n \geq 0 \text{ even} \\ 0 & \text{otherwise} \end{cases}
\end{aligned}$$

and hence the differentials are:

$$\cdots \to \mathbb{Z} \xrightarrow{\cong} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{\cong} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{0} 0$$

At degree zero, we have

$$\mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{0} 0$$

so $H_0(X) = \frac{\ker(0)}{\operatorname{im}(0)} = \frac{\mathbb{Z}}{0} = \mathbb{Z}$. At all other even degrees, we have

$$\mathbb{Z} \xrightarrow{\cong} \mathbb{Z} \xrightarrow{0} \mathbb{Z}$$

so $H_n(X) = \frac{\ker(0)}{\operatorname{im}(\cong)} = \frac{\mathbb{Z}}{\mathbb{Z}} = 0$, and at odd degrees, we have

$$\mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{\cong} \mathbb{Z}$$

so $H_n(X) = \frac{\ker(\cong)}{\operatorname{im}(0)} = \frac{0}{0} = 0$. So,

$$H_n\big(\{*\}\big) = \begin{cases} \mathbb{Z} & n = 0 \\ 0 & n \neq 0 \end{cases}$$

△

We see that, even for the most trivial of topological spaces, the singular chain complex is decidedly non-trivial. For larger spaces, the number of singular n-simplices quickly becomes unmanageable, and direct computation is rarely a feasible strategy.

Eventually, we will develop some theory that will enable the computation of singular homology groups without having to work with the singular chain complex directly, but for now, we give some interpretations of H_0 and H_1 in topological spaces.

4.1 Reduced Homology

It is often convenient for us to have a version of homology for which the one-point space has trivial homology groups in every dimension.

Let $\pi : X \to *$ be the unique morphism to the point. The *reduced homology* of X is defined as

$$\tilde{H}_n(X) := \ker\big(H_n(X) \xrightarrow{\pi_*} H_n(*)\big)$$

Equivalently, reduced homology can also be characterised as the homology of the augmented chain complex

$$\cdots \to C_2(X) \xrightarrow{\partial_2} C_1(X) \xrightarrow{\partial_1} C_0(X) \xrightarrow{\varepsilon} \mathbb{Z} \to 0$$

where $\varepsilon\left(\sum_i n_i\sigma_i\right) = \sum_i n_i$. Note that by convention, we only consider reduced homology of non-empty spaces X, or else various pathologies can arise.

Since $\varepsilon \circ \partial_1 = 0$, ε vanishes on $\operatorname{im}(\partial_1)$ and hence induces a map $H_0(X) \to \mathbb{Z}$ with kernel $\tilde{H}_0(X)$, so $H_0(X) \cong \tilde{H}_0(X) \oplus \mathbb{Z}$. Also note that, by construction, $H_n(X) \cong \tilde{H}_n(X)$ for $n > 0$.

4.2 Low-Degree Interpretation

Two n-chains x and y are *homologous* if they are in the same equivalence class or *homology class* in $H_n(X) = \frac{Z_n(X)}{B_n(X)}$, that is, if they differ by a boundary (i.e. an element of $B_n(X) = \operatorname{im}(\partial_{n+1})$) – or equivalently, if their formal difference $x - y$ or $y - x$ is itself a boundary – and we write $x \sim y$ to denote this relation.

Example. Let x and y be singular 0-simplices (points) in a space X, and suppose they lie in the same path-connected component. Let $\gamma : \Delta^1 \cong [0,1] \to X$ be a singular 1-simplex with $\gamma(0) = x$ and $\gamma(1) = y$, i.e., a path from x to y. Then,

$$\partial_1(\gamma) = y - x$$

and hence x and y are homologous, as they differ by the boundary of γ. △

Lemma 4.2. *Let X be a non-empty and path-connected* topological space. Then, $H_0(X) = \mathbb{Z}$.*

Proof. By definition,

$$H_0(X) \coloneqq \frac{Z_0\big(C_\bullet(X)\big)}{B_0\big(C_\bullet(X)\big)} = \frac{\ker(\partial_0)}{\operatorname{im}(\partial_1)}$$

The differential $\partial_0 : C_1(X) \to 0$ maps into the trivial group, so the kernel $\ker(\partial_0) = C_0(X)$ is the entire group. The idea is now to define a homomorphism from $C_0(X)$ to $\mathbb{Z}$ with kernel $B_0\big(C_\bullet(X)\big) = \operatorname{im}(\partial_1)$, which, combined with the first isomorphism theorem, the result will follow.

Define the *degree homomorphism* $\deg : C_0(X) \to \mathbb{Z}$ by sending every basis element $x \in X$ (i.e. singular 0-simplex) to $1 \in \mathbb{Z}$.

Because X is non-empty, there exists at least one basis element $x \in X$ which maps to the generator $1 \in \mathbb{Z}$, so deg is surjective.

We also have $B_0(X) \subseteq \ker(\deg)$, since the boundary $\partial_1(\gamma) \in B_0(X)$ of any singular 1-simplex has degree

$$\deg\big(\partial_1(\gamma)\big) = \deg\big(\gamma(1) - \gamma(2)\big) = 1 - 1 = 0$$

so $\partial_1(\gamma) \in \ker(\deg)$. The reverse containment $\ker(\deg) \subseteq B_0(X)$ also holds:

Let $L = \sum_{x\in X} \lambda_x \cdot x \in \ker(\deg)$ be a 0-chain whose degree vanishes. Then,

$$\begin{aligned}
L &= \sum_{x\in X} \lambda_x \cdot x \\
&= \sum_{\substack{y\in X \\ \lambda_y>0}} \lambda_y \cdot y + \sum_{\substack{z\in Z \\ \lambda_z<0}} \lambda_z \cdot z \\
&= \sum_{\substack{y\in Y \\ \lambda_y>0}} \lambda_y \cdot y - \sum_{\substack{z\in Z \\ \lambda_z<0}} (-\lambda_z) \cdot z
\end{aligned}$$

Since $\deg(L) = 0$, these two sums are equal, so we can pair up terms from each sum and write

$$L = \sum_i (y_i - z_i)$$

for some (possibly repeated) points $y_i, z_i \in X$. Since X is path-connected, there is a path γ_i from y_i to z_i for all i, so $y_i - z_i = \partial_1(\gamma_i) \in B_0(X)$, and hence $L \in B_0(X)$.

So, $\ker(\deg) = B_0(X)$. The first isomorphism theorem then gives

$$H_0(X) \coloneqq \frac{\ker(\partial_0)}{\operatorname{im}(\partial_1)} = \frac{C_0(X)}{B_0(X)} = \frac{C_0(X)}{\ker(\deg)} \cong \operatorname{im}(\deg) = \mathbb{Z}$$

as required. ∎

To interpret the 0th singular homology group for general spaces, we need the following intuitive fact:

Theorem 4.3. *Let X be a topological space and $(X_\alpha)_{\alpha\in\Lambda}$ its path-connected ccomponents. Then,*

$$H_n(X) = \bigoplus_{\alpha\in\Lambda} H_n(X_\alpha)$$

*We will assume that path-connected spaces are non-empty, and will not mention it from this point onwards.

Proof. If $\sigma : \Delta^n \to X$ is a singular n-simplex, then its image is path-connected and thus lies entirely in one of the X_α. That is, we have $C_n(X) \cong \bigoplus_{\alpha\in\Lambda} C_n(X_\alpha)$. Moreover, the oriented boundary of σ is a linear combination of $(n-1)$-simplices, all of which also lie in X_α, so ∂_n is the sum of the boundary operators for each X_α. That is, $C_\bullet(X) = \bigoplus_{\alpha\in\Lambda} C_\bullet(X_\alpha)$ as chain complexes. This decomposition therefore passes to cycles and boundaries, and eventually to homology. ■

Corollary 4.3.1. *Let X be a topological space. Then, $H_0(X) = \mathbb{Z}\pi_0(X)$. That is, $H_0(X)$ is the free abelian group with generators the path-connected components of X.*

Proof. By definition, $\pi_0(X)$ is the collection of path-connected components of X. Applying the previous two results then yields the desired result. ■

There is a similarly close relation between the first homology group $H_1(X)$ and the first homotopy group $\pi_1(X,x)$, as any path $f : [0,1] \to X$ can also be interpreted as a singular 1-simplex. In particular, if f is a loop a a path, then it is also a cycle as a singular 1-simplex, since $\partial_1(f) = f(1) - f(0) = 0$.

Lemma 4.4. *For any topological space X and all loops $f,g \in \pi_1(X,x)$,*

(i) If f is a constant path, then $f \sim 0$. That is, f is a boundary.

(ii) If $f \simeq g$, then $f \sim g$.

(iii) $f \cdot g \sim f + g$ where $\cdot$ on the left is path concatenation.

(iv) $f^{-1} \sim -f$ where f^{-1} is the reverse path of f.

Together, these properties imply that the map

$$h_1 : \pi_1(X,x) \to H_1(X)$$

defined by $h_1\big([\gamma]\big) = [\gamma]$ – where the brackets on the left mean homotopy class, and those on the right mean homology class – is a group homomorpism.

Property (i) shows that identities are mapped to identities, (ii) shows that this map is well defined, and (iii) shows that $h_1\big([f]\cdot[g]\big) = [f] + [g]$. Note, however, that this homomorphism is generally not an isomorphism, as $H_1(X)$ is abelian, while $\pi_1(X,x)$ is generally not.

For a group G, the *commutator subgroup* $[G,G]$ is the normal subgroup generated by the elements $ghg^{-1}h^{-1}$ for $g,h \in G$. The *abelianisation* of G, denoted G^{ab} is then the quotient $G/[G,G]$.

Example. If G is abelian, then $ghg^{-1}h^{-1} = \text{id}_G$ for all $g,h \in G$, so the commutator subgroup is trivial and hence $G^{\text{ab}} = G$. △

Lemma 4.5. *The abelianisation of a free product is the direct sum of the abelianisations. That is,*

$$(G * H)^{\text{ab}} \cong G^{\text{ab}} \oplus H^{\text{ab}}$$

Example. If $G = \mathbb{Z} * \mathbb{Z}$, then $G^{\text{ab}} = \mathbb{Z} \oplus \mathbb{Z}$. △

If $G = \langle S \mid R\rangle$ is a presentation, then the abelianisation is given by adjoining the commutator $[x,y]$ to the relations, for all generators $x,y \in S$.

Example. If G is given by

$$G = \langle x,y \mid x^3 = y^5\rangle$$

then the abelianisation has presentation:

$$G^{\text{ab}} = \langle x,y \mid 3x = 5y, x + y = y + x\rangle$$

Then, because x and y now commute, every element of G may be expressed in the form $ax + by$, and is equal to the identity precisely when $a = 3k$ and $b = -5k$ for some $k \in \mathbb{Z}$, so $G^{\text{ab}} \cong \mathbb{Z}^2/\mathbb{Z}(3,-5) \cong \mathbb{Z}$. △

By construction, G^{ab} is abelian, and is in fact universal with respect to this property. That is, if $\phi : G \to A$ is a morphism to an abelian group A, then there exists a unique morphism $\bar{\phi} : G^{\text{ab}} \to A$ such that the following diagram commutes:

$$\begin{tikzcd} G \arrow[r, "\phi"] \arrow[d, two heads, "\iota"'] & A \\ G^{\text{ab}} \arrow[ur, "\bar{\phi}"'] & \end{tikzcd}$$

where $\iota : G \to G^{\text{ab}}$ is the quotient map.

Lemma 4.6. *For every group homomorphism $\phi : G \to A$, $[G,G] \subseteq \ker(\phi)$.*

This gives another strategy for finding abelianisations of groups:

Example. Consider the symmetric group, S_n. The sign function $\operatorname{sgn} : S_n \to \{-1,1\} \cong \mathbb{Z}/2$ defined by

$$\sigma \mapsto \begin{cases} +1 & \sigma \text{ is even} \\ -1 & \sigma \text{ is odd} \end{cases}$$

Because $\mathbb{Z}/2$ is abelian, the commutator subgroup $[S_n,S_n]$ is contained in the kernel $\ker(\operatorname{sgn}) = A_n$. We also have $A_n \subseteq [S_n,S_n]$, since any two transpositions are conjugate in S_n, since $\sigma(i,j)\sigma^{-1} = \big(\sigma(i),\sigma(j)\big)$.

So, all transpositions are sent to the same element in $(S_n)^{\text{ab}}$. Because S_n is generated by transpositions, all non-identity elements are identified in the abelianisation, so $(S_n)^{\text{ab}} \cong \mathbb{Z}/2$. △

This universal property also implies that the map $h_1 : \pi_1(X,x) \to H_1(X)$ sending homotopy classes to homology classes factors uniquely through a morphism

$$\bar{h}_1 : \pi_1(X,x)^{\text{ab}} \to H_1(X)$$

Theorem 4.7. *For any path-connected space X, $H_1(X) \cong \pi_1(X,x)^{\text{ab}}$. More specifically, the isomorphism is given by the induced map $\bar{h}_1$.*

Corollary 4.7.1. *If X is simply connected (and hence path-connected and non-empty), then $H_1(X) = 0$.*

Intuitively, all loops in a simply connected space can contract to a point, so the space has no one-dimensional holes, and hence the first homology vanishes.

Corollary 4.7.2. *$H_1(S^1) = \mathbb{Z}$, with a generator given by the homology class of the obvious surjective map $\gamma_1 : \Delta^1 \to S^1$ identifying the end points.*

5 Fundamental Theorems

So far, we have only examined singular homology in degrees 0 and 1. For instance, we still haven't computed the higher homology groups of even basic spaces, such as the circle or higher n-spheres. As noted earlier, the singular chain complex is much too large to admit any manual computation, so here, we prove two fundamental theorems that allow us to compute the singular homology of topological spaces without directly using the singular chain complex.

5.1 Homotopy Invariance

Given a continuous map $f : X \to Y$, we can transform a singular n-simplex in X into a singular n-simplex in Y by postcomposing the singular n-simplex $\sigma : \Delta^n \to X$ by f to obtain the composition

$f \circ \sigma : \Delta^n \to Y$. We can extend this to a group homomorphism $f_\sharp : C_n(X) \to C_n(Y)$ by linearly extending

$$\begin{aligned} f_\sharp\left(\sum_i n_i \sigma_i\right) &= \sum_i n_i f_\sharp(\sigma_i) \\ &= \sum_i n_i (f \circ \sigma_i) \end{aligned}$$

How do these maps act on boundaries? Expanding the definition, we have,

$$\begin{aligned} f_\sharp(\partial(\sigma)) &= f_\sharp\left(\sum_i (-1)^i \sigma\big|_{[v_0,\dots,\widehat{v_i},\dots v_n]}\right) \\ &= \sum_i (-1)^i f_\sharp\left(\sigma\big|_{[v_0,\dots,\widehat{v_i},\dots v_n]}\right) \\ &= \sum_i (-1)^i f \circ \sigma\big|_{[v_0,\dots,\widehat{v_i},\dots v_n]} \\ &= \partial(f \circ \sigma) \\ &= \partial\big(f_\sharp(\sigma)\big) \end{aligned}$$

so, the following diagram commutes:

$$\begin{tikzcd} \cdots \arrow[r] & C_{n+1}(X) \arrow[r, "\partial"] \arrow[d, "f_\sharp"] & C_n(X) \arrow[r, "\partial"] \arrow[d, "f_\sharp"] & C_{n-1}(X) \arrow[r] \arrow[d, "f_\sharp"] & \cdots \\ \cdots \arrow[r] & D_{n+1}(X) \arrow[r, "\partial"] & D_n(X) \arrow[r, "\partial"] & D_{n-1}(X) \arrow[r] & \cdots \end{tikzcd}$$

That is, the $f_\sharp$ assemble into a chain map $f_\bullet : C_\bullet(X) \to C_\bullet(Y)$, which then induces a map in homology (Lemma 3.2).

Lemma 5.1. *Let $f : X \to Y$ be a continuous map. Then, there are induced maps in homology*

$$f_* : H_n(X) \to H_n(Y)$$

satisfying:

(*i*) $(f \circ g)_* = f_* \circ g_*$;

(*ii*) $(\mathrm{id}_X)_* = \mathrm{id}_{H_n(X)}$.

That is, $H_n(-) : \mathbf{Top} \to \mathbf{Ab}$ is a functor.

Proof. The construction is as above. Functoriality follows from the associativity of the composition $\Delta^n \xrightarrow{\sigma} X \xrightarrow{g} Y \xrightarrow{f} Z$ and the definition of an identity map. ∎

Theorem 5.2 (Homotopy Invariance)**.** *Suppose $f,g : X \to Y$ are homotopic. Then,*

$$f_* = g_* : H_n(X) \to H_n(Y)$$

Corollary 5.2.1. *If $X \simeq Y$ are homotopy equivalent, then $H_n(X) \cong H_n(Y)$ are isomorphic.*

Proof. Let $f : X \to Y$ have homotopy inverse $g : Y \to X$. Then,

$$\begin{aligned} f_* \circ g_* &= (f \circ g)_* & \qquad g_* \circ f_* &= (g \circ f)_* \\ &= (\mathrm{id}_Y)_* & &= (\mathrm{id}_X)_* \\ &= \mathrm{id}_{(H_n(X))} & &= \mathrm{id}_{H_n(X)} \end{aligned}$$

so $f_* : H_n(X) \to H_n(Y)$ and $g_* : H_n(Y) \to H_n(X)$ are inverse. ■

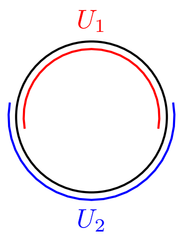
Corollary 5.2.2. *Let X be a contractible space. Then,*

$$H_n(X) = \begin{cases} \mathbb{Z} & n = 0 \\ 0 & n \neq 0 \end{cases}$$

Proof. We have previously computed the homology of a point, in this example. If X is contractible, then it is homotopy equivalent to the point, and hence has the same homology. ■

Example. We therefore know the homology of real Euclidean space $\mathbb{R}^n$, complex n-space $\mathbb{C}^n$, the unit ball D^n, the unit cube I^n, n-simplices Δ^n,etc. and every other contractible space. △

By itself, homotopy invariance is not very powerful. For instance, the sphere S^n is not contractible, nor is it homotopy equivalent to some other space whose homology we can compute (at least for $n \geq 1$). However, we can cover the sphere with (contractible) k-balls:



More generally, for any manifold X, we can find an open covering $\{U_i\}_i$ such that each U_i is homeomorphic to $\mathbb{R}^n$ for some n, and therefore contractible. If we can express the homology of X in terms of the homology of the U_i, then we could apply Homotopy Invariance. Our second fundamental theorem allows just that.

5.2 The Mayer–Vietoris Long Exact Sequence

We recall some material from homotopy theory:

Theorem (Seifert–van Kampen)**.** *Let $X = U_1 \cup U_2$ be the union of two path-connected open subspaces $j_1 : U \hookrightarrow X$, $j_2 : U_2 \hookrightarrow X$ such that $U_1 \cap U_2$ is path-connected. Then, we have a map*

$$\pi_1(U_1) * \pi_1(U_2) \xrightarrow{(j_1)_* * (j_2)_*} \pi_1(X)$$

where

1. *$(j_1)_* * (j_2)_*$ is surjective;*
2. *its kernel is the normal subgroup generated by elements of the form $i(\gamma) = (i_1)_*(\gamma)(i_2)_*(\gamma)^{-1}$, where $i_j : U_1 \cap U_2 \hookrightarrow U_j$ are the canonical inclusion maps.*

We have this setup:

$$\begin{tikzcd} U_1 \cap U_2 \arrow[r, hook, "j_2"] \arrow[d, hook, "i_1"] & U_2 \arrow[d, hook, "i_2"] \\ U_1 \arrow[r, hook, "j_1"] & X \end{tikzcd}$$

What do the induced maps in homology look like? Along the upper path, we have

$$(j_2)_* \circ (i_2)_* = (j_2 \circ i_2)_*$$

and along the lower path, we have

$$(j_1)_* \circ (i_1)_* = (j_1 \circ i_1)_*$$

by functoriality of homology. However, both maps are just the canonical inclusion of $U_1 \cap U_2$ into X, so these must be equal.

Now, passing to homology, we abelianise the groups above (Lemma 4.5 is useful here) to obtain:

$$H_1(U_1 \cap U_2) \xrightarrow{\big((i_1)_*, -(i_2)_*\big)} H_1(U_1) \oplus H_1(U_2) \xrightarrow{(j_1)_* + (j_2)_*} H_1(X)$$

where

1. $j \coloneqq (j_1)_* + (j_2)_*$ is surjective;
2. its kernel is precisely the image of $i \coloneqq \big((i_1)_*, -(i_2)_*\big)$.

To clarify how these maps act, i is a map into a product, so $i(\sigma) = \big((i_1)_*(\sigma), -(i_2)_*(\sigma)\big)$ is a pair, while $(j_1)_* + (j_2)_*$ is a map out of a product, and it acts on each component as $j(\sigma,\tau) = (j_1)_*(\sigma) + (j_2)_*(\tau)$.

The second point above is equivalent to saying that this chain complex is exact in the middle. Note that to facilitate this, we had to add a negative sign in the map i; without it, the composition would be twice the map induced by including $U_1 \cap U_2$ into X. Of course, this negative sign could be attached to any of the four maps above; it is just convention that we put it in the second component of the first map here.

If we extend the chain complex by a 0 to the right, then the first point says precisely that the chain complex is also exact at $H_1(X)$. However, as we will see, this does not typically hold when $U_1 \cap U_2$ is not path-connected.

In this, we require that U_1, U_2, and $U_1 \cap U_2$ are all path-connected, but this is rather restrictive: we still cannot apply this to the circle. However, it turns out that we can drop these assumptions in homology:

Theorem (Mayer–Vietoris Long Exact Sequence)**.** *Let X be a topologial space, and let $U_1,U_2 \subseteq X$ be not-necessarily-open subspaces whose interiors jointly cover X, and let $i_\ell : U_1 \cap U_2 \hookrightarrow U_\ell$ and $j_\ell : U_\ell \hookrightarrow X$ for $\ell = 1,2$ be the canonical inclusion maps.*

Then, there are connecting homomorphisms $\partial : H_n(X) \to H_{n-1}(U_1 \cap U_n)$ such that

$$\cdots \to H_{n+1}(X) \xrightarrow{\partial} H_n(U_1 \cap U_2) \xrightarrow{((i_1)_*, -(i_2)_*)} H_n(U_1) \oplus H_n(U_2) \xrightarrow{(j_1)_* + (j_2)_*} H_n(X) \xrightarrow{\partial} H_{n-1}(U_1 \cap U_2) \to \cdots$$

is an exact chain complex.

Recall that in a chain complex, the composition of any two maps in the sequence is the zero map. Equivalently, the image of each morphism is contained in the kernel of the next. Exactness means that the converse also holds; the image of each morphism is precisely the kernel of the next. Equivalently, its homology vanishes in each degree.

Note that, unlike for Seifert–van Kampen, the subspaces need not be open; the only requirement is that their interiors jointly cover X.

Example. Consider the circle S^1 with the same covering as previously.

The Mayer–Vietoris long exact sequence on either side of $H_n(X)$ is then

$$\cdots \to H_n(U_1) \oplus H_n(U_2) \to H_n(S^1) \xrightarrow{\partial} H_{n-1}(U_1 \cap U_2) \to \cdots$$

We will consider this situation in degrees $n \geq 2$ (so the lowest degree homology group possibly involved is $H_1(U_1 \cap U_2)$). U_1 and U_2 are contractible, so their homology is trivial, and similarly, their intersection consists of two contractible path-connected components, and homology splits across path-connected components, so the last term also vanishes, leaving:

$$\cdots \longrightarrow H_1(X) \xrightarrow{\partial} H_0(U_1 \cap U_2) \xrightarrow{\left((i_1)_*, -(i_2)_*\right)} H_0(U_1) \oplus H_0(U_2) \xrightarrow{(j_1)_* + (j_2)_*} H_0(S^1) \longrightarrow 0$$

By exactness, $H_n(S^1) = \ker(\partial) = \operatorname{im}(f) = 0$, so the homology of S^1 vanishes in degrees $n \geq 2$. For degree $n = 0$, we note that S^1 is path-connected, so $H_0(S^1) = \mathbb{Z}$; and for degree $n = 1$, we have $H_1(S^1) = \pi_1(S^1)^{\mathrm{ab}} = \mathbb{Z}$.

Overall, we have.

$$H_n(S^1) = \begin{cases} \mathbb{Z} & n = 0{,}1 \\ 0 & n \neq 0{,}1 \end{cases}$$

△

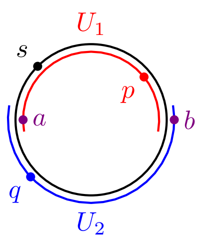
Example. We compute some of the maps at the end of the Mayer–Vietoris long exact sequence for S^1:

$$\begin{array}{ccccccccc} \cdots \longrightarrow H_1(S^1) & \xrightarrow{\partial} & H_0(U_1 \cap U_2) & \xrightarrow{\left((i_1)_*, -(i_2)_*\right)} & H_0(U_1) \oplus H_0(U_2) & \xrightarrow{(j_1)_* + (j_2)_*} & H_0(S^1) & \longrightarrow & 0 \\ & & \| & & \| & & \| & & \\ & & \mathbb{Z} \oplus \mathbb{Z} & & \mathbb{Z} \oplus \mathbb{Z} & & \mathbb{Z} & & \end{array}$$

Let us label the generators more explicitly in the long exact sequence:

$$\cdots \to H_1(S^1) \xrightarrow{\partial} \mathbb{Z}a \oplus \mathbb{Z}b \xrightarrow{\left((i_1)_*, -(i_2)_*\right)} \mathbb{Z}p \oplus \mathbb{Z}q \xrightarrow{(j_1)_* + (j_2)_*} \mathbb{Z}s \xrightarrow{0} 0$$

Because the zeroth homology groups are free abelian on the set of path-connected components, a generator is just a choice of point in each component. So, a and b are points in the intersection $U_1 \cap U_2$, with one in each component; p and q are points in U_1 and U_2, respectively; and s is some point in S^1. For instance,



The induced map $(i_1)_*$ sends the generators a and b to the generator p of $\mathbb{Z}p$, and similarly, $(i_2)_*$ sends the generators a and b to the generator q of $\mathbb{Z}q$. However, the map given by Mayer–Vietoris is given by $\left((i_1)_*, -(i_2)_*\right)$, so whenever we map into the second component, $\mathbb{Z}q$, we have a negative in the matrix:

$$\begin{array}{cc} & \begin{array}{cc} a & b \end{array} \\ \begin{array}{c} p \\ q \end{array} & \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \end{array}$$

The next map then sends both p and q to s, so the matrix is given by

$$\begin{array}{cc} & \begin{array}{cc} p & q \end{array} \\ s & \begin{bmatrix} 1 & 1 \end{bmatrix} \end{array}$$

(Again, the placement of the negative sign is arbitrary; we could have equally negated the first row of the first matrix, or either column of the second matrix, and the sequence would still be exact.) △

Corollary 5.2.3. *Let $k \geq 1$. Then,*

$$H_n(S^k) = \begin{cases} \mathbb{Z} & n = 0,k \\ 0 & n \neq 0,k \end{cases}$$

Proof. The sphere S^k can be written as the union of the upper and lower hemispheres (plus some extra space to overlap) U_1 and U_2 respectively. The Mayer–Vietoris long exact sequence is then

$$\cdots \to H_n(U_1) \oplus H_n(U_2) \to H_n(S^k) \xrightarrow{\partial} H_{n-1}(U_1 \cap U_2) \to H_{n-1}(U_1) \oplus H_{n-1}(U_2) \to \cdots$$

Each hemisphere is contractible, and the intersection $U_1 \cap U_2$ is homotopy equivalent to S^{k-1}, so in degrees $n \geq 2$, this reduces to

$$\cdots \to 0 \to H_n(S^k) \xrightarrow{\partial} H_{n-1}(S^k) \to 0 \to \cdots$$

so $H_n(S^k) \cong H_{n-1}(S^{k-1})$. We also have $H_0(S^k) \cong \mathbb{Z}$, since S^k is path-connected, and $H_1(S^k) = 0$ for $k \geq 2$, since S^k is simply connected. We have also already computed the homology for $k = 1$ in a previous example.

So, by induction on $k \geq 2$, we have

$$H_k(S^k) \underset{\cong}{\xrightarrow{\partial}} H_{k-1}(S^{k-1}) = \mathbb{Z}$$

and zero elsewhere. Along with the base cases above, this completes the proof. ■

Using reduced homology, the previous corollary becomes:

Corollary 5.2.4. *Let $k \geq 0$. Then,*

$$\tilde{H}_n(S^k) = \begin{cases} \mathbb{Z} & n = k \\ 0 & n \neq k \end{cases}$$

Theorem 5.3. *If U_1 and U_2 have non-empty intersection, then there is a sequence in reduced homology that agrees with the ordinary Mayer–Vietoris sequence in positive degrees, and ends as:*

$$\cdots \xrightarrow{\partial} \tilde{H}_0(U_1 \cap U_2) \xrightarrow{\left((i_1)_*, -(i_2)_*\right)} \tilde{H}_0(U_1) \oplus \tilde{H}_0(U_2) \xrightarrow{(j_1)_* + (j_2)_*} \tilde{H}_0(X) \xrightarrow{0} 0$$

5.3 Applications

Corollary 5.3.1. *$S^{k-1} = \partial_k D^k$ is not a retract of D^k.*

Proof. Suppose there is a retraction $r : D^k \to S^{k-1}$, so $r\big|_{S^k} = r \circ \iota = \mathrm{id}_{S^k}$. We then have the induced maps in homology $(r \circ \iota)_* = \mathrm{id}_*$:

$$\mathbb{Z} = \tilde{H}_{k-1}(S^{k-1}) \xrightarrow{\iota_*} \tilde{H}_{k-1}(D^k) \xrightarrow{r_*} \tilde{H}_{k-1}(S^{k-1}) = \mathbb{Z}$$

but D^k is contractible, so $\tilde{H}_{k-1}(D^k) = 0$, which r_* cannot surject onto $\mathbb{Z}$ from. ■

Corollary 5.3.2 (Brouwer's Fixed-Point Theorem). *Every continuous map $f : D^k \to D^k$ has a fixed point. That is, a point $x \in D^k$ such that $f(x) = x$.*

In dimension $k = 1$, this is saying that a continuous map $f : [0,1] \to [0,1]$ necessarily has a fixed point. In dimension $k = 2$, this implies that if you have a map of an area within the bounds of that area, then there is a point on that map directly above the point it represents on the Earth; this holds even if the map is folded up or crumpled into a ball, as these transformations are continuous. In dimension $k = 3$, this implies that if you stir a cup of coffee continuously, then there is a molecule whose position is unchanged after the stirring.

Proof. Suppose otherwise, that $f(x) \neq x$ for all x. For each x, consider the line connecting the distinct points $f(x)$ and x. Starting at $f(x)$ and traveling towards x, this ray intersects S^{k-1} at exactly one point x' as D^k is convex. Define a map $g : D^k \to S^{k-1}$ by $x \mapsto x'$. If x is already on the boundary, then $g(x) = x$, so g is a retraction, contradicting the previous corollary. ∎

We have another result from Brouwer:

Corollary (Invariance of Domain). *If $k \neq \ell$, then $\mathbb{R}^k \ncong \mathbb{R}^\ell$.*

Proof. Let $k \neq \ell$, and suppose $f : \mathbb{R}^k \to \mathbb{R}^\ell$ is a homeomorphism. Then, removing a point from $\mathbb{R}^k$ yields

$$S^{k-1} \simeq \mathbb{R}^k \setminus \{0\} \cong \mathbb{R}^\ell \setminus \big\{f(0)\big\} \simeq S^{\ell-1}$$

But then,

$$\mathbb{Z} = \tilde{H}_{k-1}(S^{k-1}) = \tilde{H}_{k-1}\big(\mathbb{R}^k \setminus \{0\}\big) \overset{f_*}{\cong} \tilde{H}_{k-1}\big(\mathbb{R}^\ell \setminus \{f(0)\}\big) = \tilde{H}_{k-1}(S^{\ell-1}) = 0$$

∎

6 Proof of Fundamental Theorems

6.1 Homotopy Invariance

Theorem (Homotopy Invariance). *Suppose $f,g : X \to Y$ are homotopic. Then,*

$$f_* = g_* : H_n(X) \to H_n(Y)$$

Proof sketch. Use a prism operator to construct a chain homotopy between the chain maps f_* and g_*, from which the result follows. ∎

6.1.1 Chain Homotopy

Let $a_\bullet, b_\bullet : (C_\bullet, \partial) \to (C'_\bullet, \partial')$ be two chain maps. A *chain homotopy* from $a_\bullet$ to $b_\bullet$ is a collection of morphisms

$$\eta_n : C_n \to C'_{n+1}$$

such that, in this (non-commutative!) diagram,

$$\begin{tikzcd}
\cdots \arrow[r, "\partial_{n+2}"] & C_{n+1} \arrow[r, "\partial_{n+1}"] \arrow[d, "b_{n+1}-a_{n+1}"] & C_n \arrow[r, "\partial_n"] \arrow[d, "b_n-a_n"] \arrow[dl, "\eta_n"'] & C_{n-1} \arrow[r, "\partial_{n-1}"] \arrow[d, "b_{n-1}-a_{n-1}"] \arrow[dl, "\eta_{n-1}"'] & \cdots \\
\cdots \arrow[r, "\partial'_{n+2}"] & C'_{n+1} \arrow[r, "\partial'_{n+1}"] & C'_n \arrow[r, "\partial'_n"] & C'_{n-1} \arrow[r, "\partial'_{n-1}"] & \cdots
\end{tikzcd}$$

where the red path is equal to the sum of the blue and cyan paths. That is,

$$b_n - a_n = \partial'_{n+1} \circ \eta_n + \eta_{n-1} \circ \partial_n$$

for all $n \in \mathbb{Z}$. We say that $a_\bullet$ and $b_\bullet$ are *chain homotopic* if there exists a chain homotopy between them.

Lemma 6.1. *Let $a_\bullet$ and $b_\bullet$ be chain homotopic. Then their induced maps in homology are equal:*

$$a_n = b_n : H_n(C_\bullet) \to H_n(C'_\bullet)$$

Proof. Let $c \in Z_n(C_\bullet) = \ker(\partial_n)$ be an n-cycle. Then,

$$\begin{aligned} b_n(c) - a_n(c) &= \partial'_{n+1} \circ \eta_n(c) + \eta_{n-1} \circ \partial_n(c) \\ &= \partial'_{n+1} \circ \eta_n(c) \\ &= \partial'_{n+1}\big(\eta_n(c)\big) \end{aligned}$$

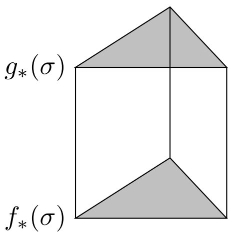
so $b_n(c) - a_n(c)$ is a boundary (of $\eta_n(c)$), so they are homologous. ■

6.1.2 Prism Operators

Given a singular n-simplex $\sigma : \Delta^n \to X$ and a homotopy $H : X \times I \to Y$ between f and g, we can compose them into a continuous map

$$H \circ (\sigma \times \mathrm{id}_I) : \Delta^n \times I \to Y$$

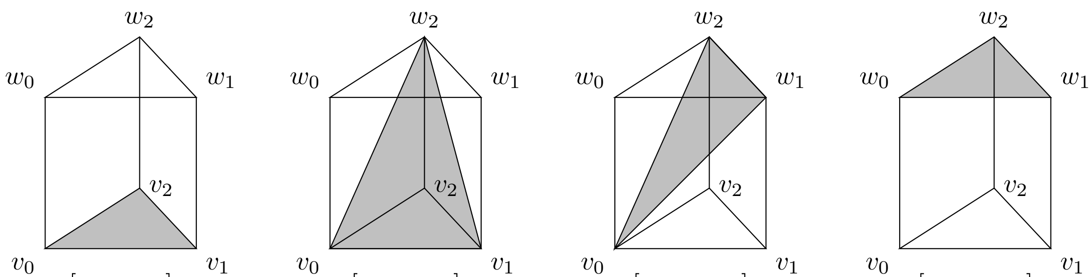
This is a homotopy from $H \circ \big(\sigma \times \{0\}\big) = f \circ \sigma = f_*(\sigma)$ to $H \circ \big(\sigma \times \{1\}\big) = g \circ \sigma = g_*(\sigma)$, which can be visualised as a prism (for $n = 2$):



The goal is now to produce an $(n+1)$-chain in Y from this data.

Denote the lower simplex by $[v_0, \ldots, v_n]$, and the upper simplex by $[w_0, \ldots, w_n]$. The idea is to generate a sequence of n-simplices that starts from the lower simplex, and ends at the upper simplex, by incrementally moving a vertex v_i up to w_i, starting with v_n, and working backwards to v_0.

So, the first step is to move $[v_0, \ldots, v_n]$ to $[v_0, \ldots, v_{n-1}, w_n]$; then the second step is to move this up to $[v_0, \ldots, v_{n-2}, w_{n-1}, w_n]$.



$[v_0,v_1,v_2]$ $[v_0,v_1,w_2]$ $[v_0,w_1,w_2]$ $[w_0,w_1,v_2]$

More generally, we move $[v_0, \ldots, v_i, w_{i+1}, \ldots, w_n]$ to $[v_0, \ldots, v_{i-1}, w_i, \ldots, w_n]$. The region between two successive n-simplices is precisely the $(n+1)$-simplex $[v_0, \ldots, v_i, w_i, \ldots, w_n]$, which has $[v_0, \ldots, v_i, w_{i+1}, \ldots, w_n]$ as its lower face, and $[v_0, \ldots, v_{i-1}, w_i, \ldots, w_n]$ as its upper face.

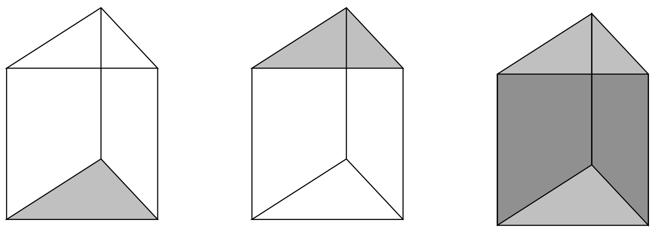
The *prism operator* $P : C_n(\Delta \times I) \to C_{n+1}(\Delta^n \times I)$ returns the alternating sum of these $(n+1)$-simplices:

$$P(\Delta^n) = \sum_{i=0}^{n} (-1)^i [v_0, \ldots, v_i, w_i, \ldots, w_n]$$

Lemma 6.2. *For all $n \geq 0$,*

$$\partial P(\Delta^n) = [w_0, \ldots, w_n] - [v_0, \ldots, v_n] - P(\partial \Delta^n)$$

Geometrically, the left side of this equation represents the boundary of the prism, while the three terms on the right represent these three pieces of the boundary:



$\Delta^n \times \{0\}$ $\Delta^n \times \{1\}$ $\partial\Delta^n \times I$

Proof of Homotopy Invariance. Let $H : X \times I \to Y$ be a homotopy from f to g, and let $\sigma : \Delta^n \to X$ be a singular n-simplex in X. This induces a map on $(n+1)$-chains

$$\big(H \circ (\sigma \times \mathrm{id}_I)\big)_* : C_{n+1}(\Delta^n \times I) \to C_{n+1}(Y)$$

Define the chain map $\eta_n : C_n(X) \to C_{n+1}(Y)$ by

$$c \mapsto \big(H \circ (c \times \mathrm{id}_I)\big)_* \big(P(\Delta^n)\big)$$

Then,

$$\begin{aligned}
\partial \eta_n(\sigma) &= \partial\big(H \circ (\sigma \times \mathrm{id}_I)\big)_* \big(P(\Delta^n)\big) \\
&= \big(H \circ (\sigma \times \mathrm{id}_I)\big)_* \big(\partial P(\Delta^n)\big) \\
&= \big(H \circ (\sigma \times \mathrm{id}_I)\big)_* \big([w_0, \ldots, w_n] - [v_0, \ldots, v_n] - P(\partial \Delta^n)\big) \\
&= g_*(\sigma) - f_*(\sigma) - \eta_{n-1}\partial(\sigma)
\end{aligned}$$

so η is a chain homotopy from f_* to g_*, so they induce equal maps in homology. ∎

6.2 Mayer-Vietoris

6.2.1 Short Exact Sequences of Chain Complexes

A *short exact sequence of chain complexes*

$$0 \to A_\bullet \xrightarrow{f_\bullet} B_\bullet \xrightarrow{g_\bullet} C_\bullet \to 0$$

is a pair of chain maps $f_\bullet$ and $g_\bullet$ such that

$$0 \to A_n \xrightarrow{f_n} B_n \xrightarrow{g_n} C_n \to 0$$

is a short exact sequence of abelian groups for all $n \in \mathbb{Z}$.

That is, the chain maps arrange into a commutative diagram

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & A_{n+1} \arrow[r, "f_{n+1}"] \arrow[d, "\partial_{n+1}"] & B_{n+1} \arrow[r, "g_{n+1}"] \arrow[d, "\partial"] & C_{n+1} \arrow[r] \arrow[d, "\partial"] & 0 \\
0 \arrow[r] & A_n \arrow[r, "f_n"] \arrow[d, "\partial"] & B_n \arrow[r, "g_n"] \arrow[d, "\partial"] & C_n \arrow[r] \arrow[d, "\partial"] & 0 \\
0 \arrow[r] & A_{n-1} \arrow[r, "f_{n-1}"] \arrow[d] & B_{n-1} \arrow[r, "g_{n-1}"] \arrow[d] & C_{n-1} \arrow[r] \arrow[d] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

where all the rows are exact and the columns are chain complexes.

Theorem 6.3. *Let*

$$0 \to A_\bullet \xrightarrow{f_\bullet} B_\bullet \xrightarrow{g_\bullet} C_\bullet \to 0$$

be a short exact sequence of chain complexes. Then, there are connecting homomorphisms $\partial : H_n(C_\bullet) \to H_{n-1}(A_\bullet)$ such that

$$\cdots \to H_{n+1}(C_\bullet) \xrightarrow{\partial} H_n(A_\bullet) \xrightarrow{f_*} H_n(B_\bullet) \xrightarrow{g_*} H_n(C_\bullet) \xrightarrow{\partial} H_{n-1}(A_\bullet) \to \cdots$$

is a long exact sequence in homology.

Proof. Let $c \in C_n$ be a cycle, i.e. $c \in \ker(\partial)$. By exactness at C_n, g_n is surjective, so $c = g_n(b)$ for some $b \in B_n$.

The element $\partial(b) \in B_{n-1}$ is in $\ker(g_{n-1})$ since, by commutativity of the lower right square, $g_{n-1}\big(\partial(b)\big) = \partial\big(g_n(b)\big) = \partial(c) = 0$. Then, by exactness at B_{n-1}, $\ker(g_{n-1}) = \operatorname{im}(f_{n-1})$, so $\partial(b) = f_n(a)$ for some $a \in A_{n-1}$.

$$\begin{tikzcd}
 & b \arrow[r, mapsto, "g_n"] \arrow[d, mapsto, "\partial"] & c \arrow[d, mapsto, "\partial"] \\
a \arrow[r, mapsto, "f_{n-1}"] & \partial(b) \arrow[r, mapsto, "g_{n-1}"] & \partial(c) = 0
\end{tikzcd}$$

We claim that a is a cycle. First, apply the differential to a, then travel along f_{n-2}:

$$\begin{tikzcd}
 & b \arrow[r, mapsto, "g_n"] \arrow[d, mapsto, "\partial"] & c \arrow[d, mapsto, "\partial"] \\
a \arrow[r, mapsto, "f_{n-1}"] \arrow[d, mapsto, "\partial"] & \partial(b) \arrow[r, mapsto, "g_{n-1}"] \arrow[d, mapsto, "\partial"] & \partial(c) = 0 \\
\partial(a) \arrow[r, mapsto, "f_{n-2}"] & f_{n-2}\big(\partial(a)\big) = \partial\big(\partial(b)\big) = 0 &
\end{tikzcd}$$

By commutativity of this lower square, $f_{n-2}\big(\partial(a)\big) = \partial\big(f_{n-1}(a)\big) = \partial\big(\partial(b)\big) = 0$, and by exactness at A_{n-2}, f_{n-2} is injective, so $\partial(a) = 0$, and a is a cycle, thus defining a homology class $[a]$.

We define the connecting homomorphism, $\partial : H_n(C) \to H_{n-1}(A)$ by $[c] \mapsto [a]$. This is well-defined since:

- The element a is uniquely determined by ∂b since f_{n-1} is injective.
- Suppose we chose a different preimage b' of c in the first step. Repeating the previous construction, we have

  $$\begin{tikzcd}
   & b' \arrow[r, mapsto, "g_n"] \arrow[d, mapsto, "\partial"] & c \\
  a' \arrow[r, mapsto, "f_{n-1}"] & \partial(b') &
  \end{tikzcd}$$

  Then, we have $g_n(b') = c = g_n(b)$, so $g_n(b' - b) = g_n(b') - g_n(b) = 0$, giving $b' - b \in \ker(g_n)$. By exactness at B_n, $\ker(g_n) = \operatorname{im}(f_n)$, so $b' - b = f_n(\sigma)$ for some $\sigma \in A_n$. Applying the differential to σ, we obtain the square:

  $$\begin{tikzcd}
  \sigma \arrow[r, mapsto, "f_n"] \arrow[d, mapsto, "\partial"] & b - b' \arrow[r, mapsto, "g_n"] \arrow[d, mapsto, "\partial"] & 0 \\
  \partial(\sigma) \arrow[r, mapsto, "f_{n-1}"] & \partial(b) - \partial(b') &
  \end{tikzcd}$$

  Now, consider $a - a'$. By construction, $f_{n-1}(a) = \partial(b)$ and $f_{n-1}(a') = \partial(b')$, so $f_{n-1}(a - a') = \partial(b) - \partial(b')$. By injectivity of f_{n-1}, $\partial(\sigma) = a - a'$. Thus, a and a' are homologous, so $[a] = [a']$.
- A different choice of c within its homology class would have the form $c + \partial(c')$ for some $c \in C_{n+1}$. By exactness at C_{n+1}, g_{n+1} is surjective, so $c' = g_n(b')$ for some $b' \in B_{n+1}$.

  $$\begin{tikzcd}
  b' \arrow[r, mapsto, "g_{n+1}"] \arrow[d, mapsto, "\partial"] & c' \arrow[d, mapsto, "\partial"] \\
  \partial(b') \arrow[r, mapsto, "g_n"] & \partial(c')
  \end{tikzcd}$$

  Then, $\partial(c') = \partial\big(g_{n+1}(b')\big) = g_n\big(\partial(b')\big)$, so $c + \partial(c') = g_n(b) + g_n\big(\partial(b')\big) = g_n\big(b + \partial(b')\big)$. So, b is replaced by $b + \partial(b')$, which leaves $\partial(b)$ and therefore also a unchanged.

This map is a group homomorphism since if $[c_1] \mapsto [a_1]$ and $[c_2] \mapsto [a_2]$ via b_1 and b_2 as above, then $g_n(b_1 + b_2) = g_n(b_1) + g_n(b_2) = c_1 + c_2$, and $f_{n-1}(a_1 + a_2) = f_{n-1}(a_1) + f_{n-1}(a_2) = \partial(b_1) + \partial(b_2) = \partial(b_1 + b_2)$, so $[c_1] + [c_2] \mapsto [a_1] + [a_2]$.

It remains to verify that the induced sequence in homology

$$\cdots \to H_{n+1}(C_\bullet) \xrightarrow{\partial} H_n(A_\bullet) \xrightarrow{f_*} H_n(B_\bullet) \xrightarrow{g_*} H_n(C_\bullet) \xrightarrow{\partial} H_{n-1}(A_\bullet) \to \cdots$$

is exact.

- $\operatorname{im}(f_*) \subseteq \ker(g_*)$: This is immediate, since $g_\bullet \circ f_\bullet = 0$ as chain maps, so $g_* \circ f_* = 0$ in homology.
- $\operatorname{im}(g_*) \subseteq \ker(\partial)$: $\partial \circ j_* = 0$ since
- $\operatorname{im}(\partial) \subseteq \ker(f_*)$
- $\ker(g_*) \subseteq \operatorname{im}(f_*)$
- $\ker(\partial) \subseteq \operatorname{im}(g_*)$
- $\ker(f_*) \subseteq \operatorname{im}(\partial)$

Omitted for now. ■

Let $U_1,U_2 \subseteq X$ be two subspaces, not necessarily open. We write $C_n(U_1 + U_2)$ for the subgroup of $C_n(X)$ consisting of n-chains that can be written as the sum of n-chains in U_1 and n-chains in U_2.

$$C_n(U_1 + U_2) \coloneqq \left\{ \sum_{i=0}^{m} \lambda_i \sigma_i : \lambda_i \in \mathbb{Z}, \sigma_i \in C_n(U_1) \cup C_n(U_2) \right\}$$

The boundary of an n-chain in U_ℓ is an $(n-1)$-chain $(n-1)$-chain in U_ℓ, so the differentials in $C_\bullet(X)$ restrict to $C_\bullet(U_1+U_2)$, so $C_\bullet(U_1+U_2)$ is a sub-chain complex. That is, the inclusion maps $C_n(U_1+U_2) \hookrightarrow C_n(X)$ define a chain map.

Theorem 6.4. *Let $j_\ell : U_\ell \hookrightarrow X$ and $i_\ell : U_1 \cap U_2 \hookrightarrow U_\ell$ be the canonical inclusion maps for $\ell = 1,2$. Then, there is a short exact sequence of chain complexes*

$$0 \to C_\bullet(U_1 \cap U_2) \xrightarrow{\left((i_1)_*, -(i_2)_*\right)} C_\bullet(U_1) \oplus C_\bullet(U_2) \xrightarrow{(j_1)_* + (j_2)_*} C_\bullet(U_1 + U_2) \to 0$$

Proof.

- The subgroup $C_n(U_1 + U_2)$ is precisely the image of $(j_1)_* + (j_2)_*$, so this map is surjective.
- It suffices to check that one of the components of $\big((i_1)_*, -(i_2)_*\big)$ is injective, and indeed, $(i_1)_*$ is an inclusion and is hence injective.
- The composition is given by

  $$\begin{aligned}\big((j_1)_* + (j_2)_*\big) \circ \big((i_1)_*, -(i_2)_*\big) &= (j_1)_*(i_1)_* - (j_2)_*(i_2)_* \\ &= (j_1 \circ i_1)_* - (j_2 \circ i_2)_* \\ &= 0\end{aligned}$$

  since $j_1 \circ i_1 = j_2 \circ i_2$ are both the inclusion $k : U_1 \cap U_2 \hookrightarrow X$. So, $\operatorname{im}(i) \subseteq \ker(j)$.

  Conversely, let $(c_1,c_2) \in \ker(j)$. That is, $j(c_1,c_2) = (j_1)_*(c_1) + (j_2)_*(c_2) = 0$, then $(j_1)_*(c_1) = -(j_2)_*(c_2)$. The left side of this equation is a chain whose simplices are contained in U_1, while the right side is a chain whose simplices are contained in U_2. So, this equality implies that all of these simplices are in the intersection $U_1 \cap U_2$, so there exists a chain $c \in C_n(U_1 \cap U_2)$ such that $k_*(c) = (j_1)_*(c_1) = -(j_2)_*(c_2)$.

Then, $(j_1)_*(c_1) = k_*(c) = (j_1)_*\big((i_1)_*(c)\big)$, so $c_1 = (i_1)_*(c)$ by injectivity of $(j_1)_*$, and similarly, $c_2 = -(i_2)_*(c)$. So, $i(c) = (c_1,c_2)$, so $\ker(j) \subseteq \operatorname{im}(i)$.

■

Theorem 6.5. *Let $U_1,U_2 \subseteq X$ be subspaces, not necessarily open. If their interiors jointly cover X, then the inclusion $\iota : C_\bullet(U_1+U_2) \hookrightarrow C_\bullet(X)$ induces isomorphisms in homology. That is,*

$$H_n\big(C_\bullet(U_1+U_2)\big) \overset{\iota_*}{\cong} H_n\big(C_\bullet(X)\big)$$

Proof of Mayer-Vietoris. By Theorem 6.3, the short exact sequence in Theorem 6.4 induces a long exact sequence in homology:

$$\cdots \to H_{n+1}\big(C_\bullet(U_1{+}U_2)\big) \xrightarrow{\partial} H_n(U_1{\cap}U_2) \xrightarrow{i} H_n(U_1){\oplus}H_n(U_2) \xrightarrow{j} H_n\big(C_\bullet(U_1{+}U_2)\big) \xrightarrow{\partial} H_{n-1}(U_1{\cap}U_2) \to \cdots$$

By the previous theorem, we may replace $H_n\big(C_\bullet(U_1+U_2)\big)$ by $H_n\big(C_\bullet(X)\big) =: H_n(X)$. ■

6.3 Barycentric subdivision

no

7 Applications

7.1 Fundamental Classes for Spheres

We have seen that the homology of the k-sphere S^k, $k \geq 1$, is given by

$$\tilde{H}_n(S^k) = \begin{cases} \mathbb{Z} & n = k \\ 0 & \text{otherwise} \end{cases}$$

A generator of $\tilde{H}_k(S^k)$ is called a *fundamental class*. Our goal is to explicitly describe these fundamental classes for all spheres by giving a cycle whose homology class is such a generator.

Example. We can view the circle S^1 as the quotient of the interval or 1-simplex. The singular 1-simplex given by the quotient map

$$\sigma : \Delta^1 \cong [0,1] \to [0,1]/0 \sim 1 \cong S^1$$

has boundary

$$\partial\sigma = * - * = 0$$

where $* : \Delta^1 \to S^1$ is the constant path at the identified point. Thus, σ is a 1-cycle which generates $H_1(S^1)$, i.e., the fundamental class of S^1. $\triangle$

However, if we try to extend this construction to S^2, we run into a problem. The 2-sphere S^2 can be obtained by quotienting the 2-simplex by its boundary. The singular 2-simplex given by the quotient map

$$\sigma : \Delta^2 \to \Delta^2/\partial\Delta^2 \cong S^2$$

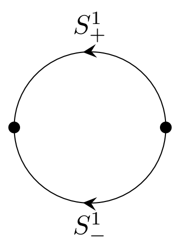
then has boundary

$$\partial\sigma = * - * + * = * \neq 0$$

so σ is not a cycle.

This construction gives a fundamental class for S^k if and only if k is odd. We describe below a construction that works in all dimensions.

Instead of taking S^1 to be the quotient of a single 1-simplex, we instead split S^1 into upper and lower hemispheres:



or,

$$S^1 \cong \frac{\Delta^1 \sqcup \Delta^1}{\sim}$$

where $\sim$ identifies the two 0th faces and two 1st faces of the two Δ^1.

Let $\sigma_+,\sigma_- : \Delta^1 \to S^1$ be the singular simplices that pick out the upper and lower hemispheres, respectively. Then, $\sigma_+ - \sigma_-$ is a cycle that represents a fundamental class.

Consider the identity map $\mathrm{id}_{\Delta^{k+1}}$ on Δ^{k+1}. This is a singular $(k+1)$-simplex in Δ^{k+1}, so we can apply the boundary operator to it:

$$\partial(\mathrm{id}_{\Delta^{k+1}}) = \sum_{i=0}^{k+1}(-1)^i[v_0,\dots,\widehat{v_i},\dots,v_n]$$

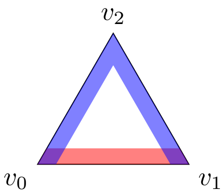
This is a linear combination of k-simplices, each in the geometric boundary $\partial\Delta^{k+1}$ of Δ^{k+1}, so it is an element of the chain group $C_{k+1}(\partial\Delta^{k+1}) \leq C_{k+1}(\Delta^{k+1})$.

Theorem 7.1. *The chain $\partial(\mathrm{id}_{\Delta^{k+1}})$ is a cycle in $C_k(\partial\Delta^{k+1})$. Moreover, it is a generator in homology.*

Proof. $\partial(\mathrm{id}_{\Delta^{k+1}})$ is already a boundary, so $\partial\big(\partial(\mathrm{id}_{\Delta^{k+1}})\big) = 0$. So, $\partial(\mathrm{id}_{\Delta^{k+1}})$ is a cycle in $C_k(\partial\Delta^{k+1})$.

Now, we induct on k. If $k = 0$, the statement is clear, so assume $k > 1$.

Let U_1 be an open neighbourhood of the last face $d_{k+1}\Delta^{k+1}$ which deformation retracts onto this face, and let U_2 be an open neighbourhood of the remaining faces $\bigcup_{0\leq i\leq k} d_i\Delta^{k+1}$ which deformation retracts onto this union. Also choose these subspaces such that their intersection $U_1 \cap U_2$ deformation retracts to the boundary of the final face $\partial d_{k+1}\Delta^{k+1} = \partial[v_0,\dots,v_k]$, and such that their union $U_1 \cup U_2$ deformation retracts to the entire boundary $\partial\Delta^{k+1} = \delta[v_0,\dots,v_{k+1}]$. One such selection is illustrated below for $k = 1$.



The Mayer–Vietoris long exact sequence for U_1 and U_2 covering $X \coloneqq U_1 \cup U - 2$ is:

$$\cdots \to \tilde{H}_k(U_1) \oplus \tilde{H}_k(U_2) \to \tilde{H}_k(U_1 \cup U_2) \xrightarrow{\partial} \tilde{H}_{k-1}(U_1 \cap U_2) \to \tilde{H}_{k-1}(U_1) \oplus \tilde{H}_{k-1}(U_2) \to \cdots$$

Because U_1 and U_2 deformation retract to a face, which is a simplex, which is contractible, the outer terms vanish, so we have an isomorphism

$$0 \to \tilde{H}_k(U_1 \cup U_2) \overset{\partial}{\cong} \tilde{H}_{k-1}(U_1 \cap U_2) \to 0$$

By the induction hypothesis, $\tilde{H}_{k-1}(U_1 \cap U_2) \cong \mathbb{Z}$, so it suffices to show that $\partial(\mathrm{id}_{\Delta^{k+1}})$ maps to a generator in $\tilde{H}_{k-1}(U_1 \cap U_2)$ under the connecting homomorphism ∂.

First, take the relevant segment of the short exact sequence of chain complexes:

$$\begin{tikzcd} & C_k(U_1) \oplus C_k(U_2) \arrow[r, "(j_1)_* + (j_2)_*"] \arrow[d, "\partial"] & C_k(U_1 + U_2) \\ C_{k-1}(U_1 \cap U_2) \arrow[r, "\big((i_1)_*, -(i_2)_*\big)"] & C_{k-1}(U_1) \oplus C_{k-1}(U_2) & \end{tikzcd}$$

starting with $\partial(\mathrm{id}_{\Delta^{k+1}}) \in C_k(U_1 + U_2)$. This cycle has a lift in $C_k(U_1) \oplus C_k(U_2)$, given by

$$\left((-1)^{k+1}[v_0,\dots,v_k], \sum_{i=1}^{k}(-1)^i[v_0,\dots,\widehat{v_i},\dots,v_{k+1}]\right)$$

which maps down to

$$\left((-1)^{k+1}\partial\big([v_0,\dots,v_k]\big), \sum_{i=1}^{k}(-1)^i\partial\big([v_0,\dots,\widehat{v_i},\dots,v_{k+1}]\big)\right)$$

So, we are looking for the unique $(k-1)$-cycle σ in $U_1 \cap U_2$ satisfying

$$\big((i_1)_*(\sigma), -(i_2)_*(\sigma)\big) = \left((-1)^{k+1}\partial\big([v_0,\dots,v_k]\big), \sum_{i=1}^{k}(-1)^i\partial\big([v_0,\dots,\widehat{v_i},\dots,v_{k+1}]\big)\right)$$

By comparing the first components, it is clear that $\sigma = (-1)^{k-1}\partial\big([v_0,\dots,v_k]\big)$ works. ■

Let S^k_+ and S^k_- be the upper and lower hemispheres of S^k. Choose homeomorphisms

$$\sigma_+ : \Delta^k \xrightarrow{\cong} S^k_+ \qquad \sigma_- : \Delta^k \xrightarrow{\cong} S^k_-$$

such that

- σ_+ and σ_- both map the boundary $\partial\Delta^k$ homeomorphically onto the equator $S^k_+ \cap S^k_-$;
- the composition $\partial\Delta^k \xrightarrow{\sigma_+} S^k_+ \cap S^k_- \xrightarrow{(\sigma_-)^{-1}} \partial\Delta^k$ is the identity.

Corollary 7.1.1. *The chain $\sigma_+ - \sigma_- \in C_k(S^k)$ is a cycle, and represents a fundamental class for S^k.*

Proof. We have seen this for $k = 0{,}1$ above, so assume $k \geq 2$.

The second requirement on σ_+ and σ_- say that their boundaries are the same, so $\partial(\sigma_+ - \sigma_-) = \partial(\sigma_+) - \partial(\sigma_-) = 0$, so $\sigma_+ - \sigma_-$ is a cycle.

Now, choose open neighbourhoods U_+ and U_- of the two hemispheres which deformation retract to the hemispheres, and whose intersection $U_- \cap U_+$ deformation retracts onto the equator. Then, the Mayer–Vietoris long exact sequence for U_1 and U_2 covering $S^k = U_1 \cup U_2$ is:

$$\cdots \to H_k(U_+) \oplus H_k(U_-) \to H_k(S^k) \to H_{k-1}(U_+ \cap U_-) \to H_{k-1}(U_+) \oplus H_{k-1}(U_-) \to \cdots$$

The subspaces are contractible, so their homology vanishes, leaving an isomorphism

$$0 \to H_k(S^k) \overset{\partial}{\cong} H_{k-1}(U_+ \cap U_-) \to 0$$

So, it suffices to prove that $\sigma_+ - \sigma_- \in H_k(S^k)$ maps to a generator under the connecting homomorphism. Again, we take the relevant segment of the short exact sequence of chain complexes:

$$\begin{tikzcd}
 & C_k(U_-) \oplus C_k(U_+) \arrow[r, "(j_1)_* + (j_2)_*"] \arrow[d, "\partial"] & C_k(U_+ + U_-) \\
C_{k-1}(U_+ \cap U_-) \arrow[r, "\big((i_1)_*, -(i_2)_*\big)"] & C_{k-1}(U_+) \oplus C_{k-1}(U_-) &
\end{tikzcd}$$

and chase $\sigma_+ - \sigma_-$

$$\begin{tikzcd}
 & (\sigma_+, -\sigma_-) \arrow[r, mapsto, "(j_1)_* + (j_2)_*"] \arrow[d, mapsto, "\partial"] & \sigma_+ - \sigma_- \\
\partial(\sigma_+) = \partial(\sigma_-) \arrow[r, mapsto, "\big((i_1)_*, -(i_2)_*\big)"] & \big(\partial(\sigma_+), -\partial(\sigma_-)\big) &
\end{tikzcd}$$

So, the connecting homomorphism maps $\sigma_+ - \sigma_-$ to $\partial(\sigma_+)$. Then, by construction, $U_+ \cap U_- \simeq S^k_+ \cap S^k_- \overset{\sigma_+}{\cong} \partial\Delta^k$, and $\partial(\mathrm{id}_{\Delta^k})$ is a generator of $H_{k-1}(\partial\Delta^k)$. ■

7.2 Jordan Curve Theorem

Recall that a *Jordan curve* is a simple closed curve in $\mathbb{R}^2$. Informally, the *Jordan curve theorem* states that

> Every jordan curve splits the plane into two regions.

One of the two regions is necessarily bounded and is thus interpreted as the *interior*, while the other region is necessarily unbounded and is thus interpreted as the *exterior*. The Jordan curve is then the boundary of each of these regions.

This is intuitively clear for any reasonably nice curve, but is difficult to interpret for, say, fractal curves.

Theorem 7.2 (Jordan Curve Theorem)**.** *Let $\gamma : S^1 \rightarrowtail \mathbb{R}^2$ be an injective continuous map with image $C \subseteq \mathbb{R}^2$. Then,*

$$H_n(\mathbb{R}^2 \setminus C) = \begin{cases} \mathbb{Z}^2 & n = 0 \\ \mathbb{Z} & n = 1 \\ 0 & n > 1 \end{cases}$$

Because $\mathbb{R}^2 \setminus C$ is locally path-connected, the part $H_0(\mathbb{R}^2 \setminus C) = \mathbb{Z}^2$ is saying precisely that the complement of C has two path-connected components.

We translate the problem as follows: let $\mathbb{R}^2 \hookrightarrow \mathbb{R}^2 \cup \{\infty\} \cong S^2$ be the one-point compactification of $\mathbb{R}^2$. Then,

$$H_n(S^2 \setminus C) = \begin{cases} \mathbb{Z}^2 & n = 0 \\ 0 & n > 0 \end{cases}$$

To prove this, we need a few more lemmata.

Lemma 7.3. *Let $\kappa : [0,1] \rightarrowtail S^2$ be an injective continuous map with image $D \subseteq \mathbb{R}^2$. Then,*

$$H_n(S^2 \setminus D) = \begin{cases} \mathbb{Z} & n = 0 \\ 0 & n > 0 \end{cases}$$

Proof of Jordan Curve Theorem. We compute the homology of $S^2 \setminus C$. Let S^1_+ and S^1_- be the upper and lower semicircles in S^1 such that $S^1_+ \cap S^1_- = S^0$. Now, apply Mayer–Vietoris in reduced homology with

- $U_+ \coloneqq S^2 \setminus \gamma(S^1_+)$;
- $U_- \coloneqq S^2 \setminus \gamma(S^1_-)$;
- $X \coloneqq U_+ \cup U_2 = S^2 \setminus \gamma(S^0)$;
- $U_+ \cap U_1 = S^2 \setminus C$.

As S^1_+ and S^1_- are homeomorphic to $[0,1]$, we have the homology groups of U_+ and U_- from the previous lemma, and in reduced homology, these vanish in all degrees. Also, X is the twice-punctured 2-sphere – the punctured 2-sphere is homeomorphic to the plane $\mathbb{R}^2$, and the punctured plane is homotopy equivalent to the circle S^1 – so $X \simeq S^1$, and these homology groups also vanish in degrees $n > 1$.

So, the long exact sequence is zero everywhere past $H_1(X)$. The end of the sequence is then given by:

$$\cdots \to 0 \to \tilde{H}_1(X) \to \tilde{H}_0(U_+ \cap U_-) \to 0 \oplus 0 \to 0 \to 0$$

so $\tilde{H}_0(S^2 \setminus C) = \tilde{H}_0(U_+ \cap U_-) \cong \tilde{H}_1(X) = \mathbb{Z}$. So, $H_0(S^2 \setminus C) = \mathbb{Z}^2$. ■

7.3 Relative Homology

Let $A \subseteq X$ be a subspace, and $\iota : A \hookrightarrow X$ be the canonical inclusion map. Then, there is an induced inclusion between chain groups, $C_n(A) \hookrightarrow C_n(X)$, and these inclusions assemble into a chain map $C_\bullet(A) \to C_n(X)$.

However, the induced map in homology $\iota_* : H_n(A) \to H_n(X)$ is not injective in general (nor surjective). For instance, if $A = S^1$ and $X = D^2$, then $H_1(A) = \mathbb{Z}$ cannot inject into $H_1(X) = 0$.

We define the group of *relative singular n-chains* $C_n(X,A)$ as the quotient $C_n(X)/C_n(A)$.

$$C_n(A) \hookrightarrow C_n(X) \to \frac{C_n(X)}{C_n(A)} =: C_n(X,A)$$

Because $C_\bullet(A)$ is a sub-chain complex of $C_\bullet(X)$, $C_\bullet(X,A)$ also inherits the structure of a chain complex called the *relative singular chain complex*, with the differentials $\partial : C_n(X,A) \to C_{n-1}(X,A)$ induced from $\partial : C_n(X) \to C_{n-1}(X)$. The quotient maps $C_n(X) \to C_n(X,A)$ also assemble into a chain map $C_\bullet(X) \to C_\bullet(X,A)$.

The *relative homology* of the pair (X,A) is then given by the homology of the relative singular chain complex:

$$H_n(X,A) := H_n\big(C_\bullet(X,A)\big)$$

- Call an n-chain $c \in C_n(X)$ a *relative n-cycle* if $\partial(c) \in C_{n-1}(A)$. For example, a singular n-simplex $\sigma : \Delta^n \to X$ is a relative n-cycle if the image of the boundary $\partial\Delta^n$ is contained in A.
- Call an n-chain $c \in C_n$ a *relative n-boundary* if it is homologous to some n-chain in A. That is, if there exists $a \in C_n(A)$ such that the difference $c - a = \partial w$ is the boundary of some $(n+1)$-chain $w \in C_{n+1}(X)$. Note that every relative n-boundary is a relative n-cycle since $\partial c = \partial a$.

By construction,

$$H_n(X,A) \cong \frac{\text{relative } n\text{-cycles}}{\text{relative } n\text{-boundaries}}$$

Intuitively, the relative homology $H_n(X,A)$ measures the homology of X with A "discarded".

Corollary 7.3.1. *There is an exact sequence in homology:*

$$\begin{tikzcd}
\cdots \arrow[r, "\partial"] & H_n(A) \arrow[r] & H_n(X) \arrow[r] & H_n(X,A) \arrow[dll, "\partial"] & \\
& H_{n-1}(A) \arrow[r] & H_{n-1}(X) \arrow[r] & H_{n-1}(X,A) \arrow[r, "\partial"] & \cdots
\end{tikzcd}$$

Proof. By construction, there is a short exact sequence of chain complexes

$$0 \to C_\bullet(A) \to C_\bullet(X) \to C_\bullet(X,A) \to 0$$

so the claim follows from Theorem 6.3. ■

We can also describe the connecting homomorphism explicitly:

If $[z] \in H_n(X,A)$ is represented by a relative cycle $z \in C_n(X)$, then the connecting homomorphism is defined by

$$\partial[z] = [\partial z]$$

Because z is a relative cycle, its boundary ∂z is contained in A, so this class $[\partial z]$ is an element of $H_{n-1}(A)$.

Theorem 7.4 (Excision)**.** *Let $Z \subseteq A \subseteq X$, with $\bar{Z} \subseteq A^\circ$. Then,*

$$H_n(X,A) \cong H_n(X \setminus Z, A \setminus Z)$$

Intuitively, the relative homology ignores the interior of A, so we may excise a portion Z, with minor restrictions.

Recall that a *topological manifold of dimension k* is a Hausdorff space such that every point has an open neighrbourhood homeomorphic to $\mathbb{R}^k$. Every smooth manifold is a topological manifold.

Corollary 7.4.1. *Let M be a k-dimensional topological manifold and let $x \in M$ be a point. Then,*

$$H_n(M, M \setminus x) \cong H_n(\mathbb{R}^k, \mathbb{R}^k \setminus *) \cong \begin{cases} \mathbb{Z} & n = k \\ 0 & n \neq k \end{cases}$$

That is, relative singular homology is able to detect the dimension of a manifold.

Proof. Let $U \ni x$ be an open neighbourhood of x homeomorphic to $\mathbb{R}^k$. Then, Excision gives the first isomorphism with $X = M$, $A = M \setminus x$, and $Z = M \setminus U$. For the second isomorphism, consider the long exact sequence of the pair $\left(\mathbb{R}^k, \mathbb{R}^k \setminus *\right)$. ■

Corollary 7.4.2 (Invariance of Domain II)**.** *Let $U \subseteq \mathbb{R}^k$ and $V \subseteq \mathbb{R}^\ell$ be non-empty open subsets. If $U \cong V$, then $k = \ell$.*

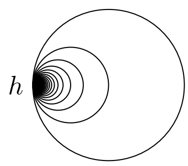
A pair (X,A) is *good* if:

(*i*) $A \subseteq X$ is closed;

(*ii*) there exists an open neighbourhood $V \supseteq A$ which deformation retracts onto A.

Example. If X is a CW-complex, then (X,A) is good for any subcomplex A. △

Example. Consider the Hawaiian earring H with $h \in H$ the distinguished point where all the circles meet.



Then, (H,h) is not a good pair, since any open neighbourhood of h contains infinitely many circles and cannot be contractible △

Example. Let $X = [0,1]$ be the interval, and $A = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots, 0\right\} \subseteq X$.

(X,A) is not a good pair, because $(X/A, A/A)$ would then also be a good pair. But $(X/A, A/A) \cong (H,h)$. △

Theorem 7.5. *Let (X,A) be a good pair. Then, the quotient map $X \to X/A$ induces isomorphisms*

$$H_n(X,A) \cong H_n(X/A, A/A) \cong \tilde{H}_n(X/A)$$

Example. $(\Delta^k, \partial\Delta^k)$ is a good pair for any k, so

$$H_n(\Delta^k, \partial\Delta^k) \cong \tilde{H}_n(S^k) \cong \begin{cases} \mathbb{Z} & n = k \\ 0 & n \neq k \end{cases}$$

△

8 Degrees

Let $f : S^k \to S^k$ be a continuous map. Then, the induced map in kth homology,

$$f_* : \tilde{H}_k(S^k) \to \tilde{H}_k(S^k)$$

is a group homomorphism $\mathbb{Z} \to \mathbb{Z}$. Such a homomorphism is determined entirely by the image of the generator $1 \mapsto d$, and thus acts by multiplication by d. This integer d is called the *degree* of f, written as $\deg(f)$.

Lemma 8.1. *Let $f,g : S^k \to S^k$. Then,*

(*i*) $\deg(\mathrm{id}_{S^k}) = 1$;

(*ii*) $\deg(g \circ f) = \deg(g) \cdot \deg(f)$;

(*iii*) *if $f \simeq g$, then* $\deg(f) = \deg(g)$;

(*iv*) *if f is a homotopy equivalence, then* $\deg(f) = \pm 1$;

(*v*) *if f is not surjective, then* $\deg(f) = 0$.

Proof.

(*i*) The identity induces the identity in homology.

(*ii*) $(g \circ f)_* = g_* \circ f_*$.

(*iii*) By homotopy invariance, f and g induce the same maps in homology, so they have the same degree.

(*iv*) if f is a homotopy equivalence, then it induces an isomorphism in homology. The only possible images for the generator 1 are then the generators 1 and -1.

(*v*) Let $x \in S^k$ be outside the image of f. Then, f factors as

$$\begin{tikzcd} S^k \arrow[rr, "f"] \arrow[dr, "f"'] & & S^k \\ & S^k \setminus x \arrow[ur, hook, "\iota"'] & \end{tikzcd}$$

Then in reduced homology, f_* factors through $\tilde{H}_k(S^k \setminus x) = 0$ since $S^k \setminus x$ is contractible.

$$\begin{tikzcd} \mathbb{Z} \arrow[rr, "f_*"] \arrow[dr] & & \mathbb{Z} \\ & 0 \arrow[ur] & \end{tikzcd}$$

so $\deg(f) = 0$.

■

Example. Consider an endomorphism on the 0-sphere:

$$\begin{array}{ccc} S^0 & \xrightarrow{f} & S^0 \\ \cong & & \cong \\ \{a,b\} & & \{a,b\} \end{array}$$

There are only four possible maps:

$$(a,b) \mapsto \begin{cases} (a,a) \\ (b,b) \\ (a,b) \\ (b,a) \end{cases}$$

The first two maps are not surjective, so they have degree $\deg(f) = 0$. The third map is the identity, so in this case, $\deg(f) = 1$. For the final map, consider the reduced homology:

$$\begin{aligned} \tilde{H}_n(S^0) &= \ker\Big(H_0(S^0) \xrightarrow{\pi} H_0(*)\Big) \\ &= \ker\Big(\mathbb{Z}a \oplus \mathbb{Z}b \xrightarrow{\pi} \mathbb{Z}*\Big) \end{aligned}$$

$a - b$ is a generator of $\tilde{H}_0(S^0)$ since $\pi(a-b) = * - * = 0$, so $\tilde{H}_0(S^0) = \mathbb{Z}(a-b)$. Then,

$$f_*(a-b) = b - a = -(a-b)$$

so $\deg(f) = -1$. △

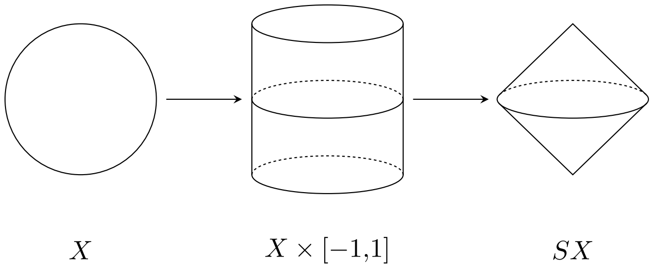
Example. Consider S^1 as a subset of the complex numbers, and let $f : S^1 \to S^1$ be defined by $z \mapsto z^n$ for some integer n. The loop $\sigma : [0,1] \to S^1$ defined by $t \mapsto e^{2\pi i t}$ represents a generator of $H_1(S^1) \cong \tilde{H}_1(S^1)$. Then, $f_*\big([\sigma]\big) = [f \circ \sigma]$ is represented by the loop $t \mapsto e^{2\pi i n t}$ which is homologous to $n\sigma$, so $\deg(f) = n$. △

Theorem 8.2. *Let $k \geq 1$. For every integer $n \in \mathbb{Z}$, there exists a map $f : S^k \to S^k$ of degree n.*

Proof. The *suspension* SX of a space X is the space

$$X \times [-1,1]\big/$$

where $(x,1) \sim (y,1)$ and $(x,-1) \sim (y,-1)$ for all $x,y \in X$.



Taking the upper and lower cones, plus some extra space for overlap:

$$C_+X := X \times (-\varepsilon,1]\big/X \times 1 \qquad C_-X := X \times [-1,\varepsilon)\big/X \times -1$$

we have two open subspaces that jointly cover X, and their intersection deformation retracts to X. These subspace are also contractible, so their homology vanishes, and Mayer–Vietoris gives an isomorphism

$$H_{k+1}(SX) \overset{\partial}{\cong} H_k(X)$$

Then, a map $f : X \to Y$ induces a map $Sf : SX \to SY$, and we have a commutative square

$$\begin{array}{ccc} H_{k+1}(SX) & \xrightarrow{\partial} & H_k(X) \\ \downarrow{\scriptstyle Sf_*} & & \downarrow{\scriptstyle f_*} \\ H_{k+1}(SY) & \xrightarrow{\partial} & H_k(Y) \end{array}$$

Applying this with $X = Y = S^{k-1}$, and noticing the suspension of S^{k-1} is homeomorphic to S^k, we have $deg(Sf) = \deg(f)$. We can then reduce inductively to $k = 1$. ■

8.1 Antipodes

Lemma 8.3. *Let $S^k \subseteq \mathbb{R}^{k-1}$ be the unit circle. Let $f : S^k \to S^k$ be the reflection in a hyperplane through the origin. Then* $\deg(f) = -1$.

Proof. Let $H \subseteq \mathbb{R}^{k-1}$ be the fixed hyperplane. It splits the sphere S^k into two hemispheres S^k_+ and S^k_-. Fix some homeomorphism $\sigma_+ : \Delta^k \to S^k_+$, and set $\sigma_- = f \circ \sigma_+$.

Because f is the identity on H, the composition

$$\partial\Delta^k \xrightarrow{\sigma_+} S^k_+ \cap S^k_- \xrightarrow{(\sigma_-)^{-1}} \partial\Delta^k$$

is the identity:

$$\begin{aligned}(\sigma_-)^{-1} \circ \sigma_+ &= (f \circ \sigma_+)^{-1} \circ \sigma_+ \\ &= (\sigma_+)^{-1} \circ \sigma_+ \\ &= \mathrm{id}_{\partial\Delta^k}\end{aligned}$$

so Corollary 7.1.1 applies. So, $[\sigma_+ - \sigma_-]$ generates $\tilde{H}_k(S^k)$, and

$$f_*\big([\sigma_+ - \sigma_-]\big) = [f \circ \sigma_+] - [f \circ \sigma_-] = [\sigma_-] - [\sigma_+] = -[\sigma_+ - \sigma_-]$$

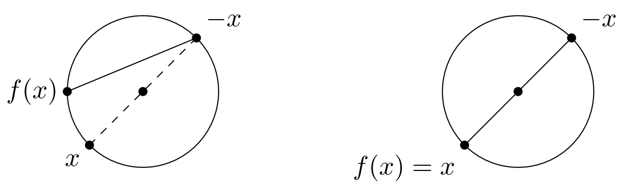
so $\deg(f) = -1$. ■

Theorem 8.4. *Let $T : \mathbb{R}^{k+1} \to \mathbb{R}^{k+1}$ be a orthogonal linear transformation. It restricts to a homeomorphism $f : S^k \to S^k$. Then* $\deg(f) = \det(T)$.

Corollary 8.4.1. *Let $f : S^k \to S^k$ be the antipodal map $x \mapsto -x$. Then,* $\deg(f) = (-1)^{k+1}$.

Corollary 8.4.2. *If $f : S^k \to S^k$ has no fixed points, then* $\deg(f) = (-1)^{k+1}$.

Proof. We show that f is homotopic to the antipodal map. The line through $f(x)$ and $-x$ passes through the origin if and only if $f(x)$ and $-x$ are antipodal. That is, if $f(x) = x$.



Since f has no fixed points, this cannot be the case, so the line $tf(x) + (1-t)(-x)$ connecting the two points, parametrised by t, is never zero. So, dividing by its norm yields an element of S^1, so

$$(t{,}x) \mapsto \frac{tf(x) + (1-t)(-x)}{\|tf(x) + (1-t)(-x)\|}$$

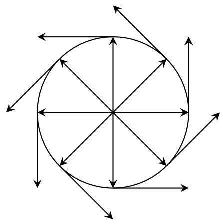
is a homotopy from the antipodal map to f. ■

Recall that a vector field on S^k is a continuous map $v : S^k \to \mathbb{R}^{k+1}$. A vector field is a *tangent vector field* if $v(x)$ is orthogonal to x for all $x \in S^k$.

Example. The constant map $v(x) = 0$ is a tangent vector field on every sphere since the zero vector is orthogonal to every vector. $\triangle$

We are interested in tangent vector fields that vanish nowhere.

Example. On the 1-sphere,



is a non-vanishing tangent vector field. This construction generalises to other odd-dimensional spheres as:

$$x = (x_1, \ldots ,x_{2m}) \mapsto v(x) - (-x_2,x_1, - x_4,x_3, \ldots, - x_{2m},x_{2m-1})$$

$\triangle$

Do there exist non-vanishing tangent vector fields on *even*-dimensional spheres?

Corollary 8.4.3 (Hairy Ball Theorem)**.** *Every tangent vector field on an even-dimensional sphere vanishes at some point.*

Proof. Suppose for a contradiction that $v : S^k \to \mathbb{R}^{k+1}$ is a non-vanishing tangent vector field with k even. Because $v(x)$ and x are non-zero and orthogonal, they are linearly independent, so the map

$$S^k \times [0,1] \ni (x,t) \mapsto \cos(\pi t)x + \sin(\pi t)v(x) \in \mathbb{R}^{k+1}$$

cannot vanish. So, we can divide by its norm to obtain a homotopy $S^k \times [0,1] \to S^k$ from the identity, at $t = 0$, to the antipodal map, at $t = 1$.

But, this is impossible, as identity has degree 1, while the antipodal map has degree $(-1)^{k+1} = -1$. $\blacksquare$

8.2 Local Degrees

Let $k \geq 1$, and let $f : S^k \to S^k$ be a continuous map, and let $y \in S^k$ such that its preimage $f^{-1}(y) = \{x_1, \ldots ,x_n\}$ is finite. We may choose disjoint open balls $U_i \subseteq S^k$ around the x_i.

So, f induces a map of pairs $(U_1,U_1 \setminus x_i) \hookrightarrow (S^k,S^k \setminus y)$, and hence by excision, a map in homology

$$H_k(S^k,S^k \setminus x_i) \cong H_k(U_i,U_i \setminus x_i) \xrightarrow{f_*} H_k(S^k,S^k \setminus y)$$

Now, recall the long exact sequence in relative homology:

$$\cdots \to H_n(A) \to H_n(X) \to H_n(X,A) \to H_{n-1}(A) \to \cdots$$

On the left, we have $A = S^k \setminus x_i$, which is contractible, so the outer terms vanish and we have an isomorphism

$$H_k(S^k) \cong H_k(S^k,S^k \setminus x_i)$$

Similarly, on the right we have $A = S^k \setminus y$, which is also contractible, so we again have an isomorphism

$$H_k(S^k) \cong H_k(S^k,S^k \setminus y)$$

So, the whole composition

$$\begin{tikzcd} H_k(S^k,S^k \setminus x_i) \arrow[r, "\cong"] & H_k(U_i,U_i \setminus x_i) \arrow[r, "f_*"] & H_k(S^k,S^k \setminus y) \\ H_k(S^k) \arrow[u, "\cong"] \arrow[rr, "f\big|_{x_i}"] & & H_k(S^k) \arrow[u, "\cong"] \end{tikzcd}$$

can be viewed as an endomorphism

$$f\big|_{x_i} : H_k(S^k) \to H_k(S^k)$$

for each x_i. $H_k(S^k) \cong \mathbb{Z}$, so these maps are given by multiplication by some integer, called the *local degree* of f at x_i, denoted by $\deg(f|_{x_i})$.

Theorem 8.5. *In the situation above,*

$$\deg(f) = \sum_{i=1}^{n} \deg(f|_{x_i})$$

Example. Let $p(z) \in \mathbb{C}[z]$ be a non-constant polynomial interpreted as a map $\mathbb{C} \to \mathbb{C}$. It extends to a continuous map on the one-point compactification

$$f_p : S^2 \cong \mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\} \cong S^2$$

Let $w \in \mathbb{C}$ be such that $p'(z_i) \neq 0$ for all $z_i \in f_p^{-1}\big[\{w\}\big]$. Such a w always exists as p' vanishes at finitely many points. So, f_p is invertible around each z_i, so $\deg(f_p|_{z_i}) = \pm 1$. In fact, since polynomials are orientation preserving, we must have $\deg(f_p|_{z_i}) = 1$. So,

$$\deg(f_p) = \sum_{i=1}^{n} \deg(f_p|_{z_i}) = n = \deg(p)$$

△

9 Manifolds

We have already recalled the notion of a (topological) manifold: a Hausdorff space such that every point has an open neighbourhood homeomorphic to $\mathbb{R}^k$, for some fixed k called the *dimension* of the manifold.

Example.

- Euclidean space $\mathbb{R}^k$ is itself a k-manifold.
- The sphere S^k is a k-manifold: for any point that isn't the north pole, take the open neighbourhood to be the entire sphere minus the north pole, which is homeomorphic to $\mathbb{R}^k$ via sterographic projection. For the north pole, take the open neighbourhodo to be the entire sphere minus the south pole, which is again homeomorphic to $\mathbb{R}^k$ via sterographic projection.
- Any open subspace of a k-manifold is itself a k-manifold.
- The torus $\mathbb{T}$, Klein bottle $\mathbb{K}$, and $\mathbb{RP}^2$ are all 2-manifolds.

△

Example. The interval [0,1] is not a manifold, since no open neighbourhood of 0 or 1 is homeomorphic to $\mathbb{R}^k$ for any k. (Instead, it is a *manifold with boundary*, which we will not discuss.) △

Example. A 0-manifold is any space with the discrete topology (i.e. every set is open): for an open neighourhood of a point x to be homeomorphic to $\mathbb{R}^0 \cong \{*\}$, it must be a singleton set, namely $\{x\}$, so the topology is discrete. △

Theorem 9.1. *The only connected compact* 1*-manifold is* S^1*.*

Recall that a *covering* of a topological space X is a map $p : \tilde{X} \to X$ such that for every point $x \in X$, there exists an open neighbourhood $U_x \subseteq X$ of x whose preimage

$$p^{-1}[U_x] = \bigsqcup_{i \in I_x} V_i$$

is a disjoint union of open sets $(V_i)_{i \in I_x}$, and the restriction $p\big|_{V_i} : V_i \to U_x$ is a homeomorphism for every $i \in I_x$. Such an open set U_x is said to be *evenly covered* by p, and the open sets V_i are called the *sheets* of the covering. If $p : \tilde{X} \to X$ is a covering, then the pair $(\tilde{X}, p)$ is called a *covering space* or *cover* of X, and X is said to be the *base* of the covering.

If X is connected, then the indexing set I_x does not depend on X. If $|I| = n$, then we say that $p : \tilde{X} \to X$ is an *n-fold* or *n-sheeted* cover.

Intuitively, a covering is a surjective map that acts locally like a projection of multiple copies of a space onto itself.

Example. For any $k \in \mathbb{N}$, the map $p_k : S^1 \to S^1$ defined by $z \mapsto z^k$ is a covering map. The preimage of the arc of length $\frac{1}{k}$ centred on z is the collection of arcs that each cover $\frac{1}{k}$th of the circle, centred on each root of z, and these arcs are disjoint as there are exactly k such roots evenly spaced along the circle.

This covering is also an k-fold covering map, as the fibre of any point $z = \exp(2\pi i t)$ consists of k many kth roots of z – namely $\exp\bigl(2\pi i(t + j)/k\bigr)$, for $0 \le j < k$. △

Example. The map $p_\infty : \mathbb{R} \to S^1$ defined by $x \mapsto \exp(2\pi i x)$ is a covering map. Given a point $z = \exp(2\pi i t) \in S^1$, we take the open neighbourhood $U = \bigl\{\exp(2\pi i s) : |s - t| < \varepsilon\bigr\}$ for some $0 < \varepsilon < 1$, which has preimage

$$\begin{aligned} p^{-1}[U] &= \bigcup_{j \in \mathbb{Z}} \{s + i : |s - t| < \varepsilon\} \\ &= \bigsqcup_{j \in \mathbb{Z}} V \end{aligned}$$

△

Lemma 9.2. *Let X be a k-manifold, and let $p : Y \to X$ be a covering space. Then, Y is also a k-manifold.*

Proof. Let $y_1, y_2 \in Y$ be distinct points, with images $x_1 = p(y_1) = x_1$ and $x_2 = p(y_2)$. If $x_1 \neq x_2$, then y_1 and y_2 are in different fibres: there exist disjoint open neighbourhoods $U_i \subseteq X$ of x_i since X is Hausdorff; then, $V_i \coloneqq p^{-1}[U_i]$ are disjoint open neighbourhoods of the y_i.

If $x_1 = x_2$, then y_1 and y_2 are in the same fibre, but different sheets, since $y_1 \neq y_2$ and coverings are homeomorphic on sheets: choose an evenly covered neighbourhood $U \subseteq X$ of $x_1 = x_2$; then, y_1 and y_2 are in disjoint open sheets of U.

So, Y is Hausdorff.

Let $y \in Y$ have image $x = p(y)$. By assumption, there exists an evenly covered neighbourhood $U \subseteq X$ of x, so y is in a sheet V_y of U. Since U is open in X, it is also a manifold, so there is an open neighbourhood $V \subseteq U$ of x homeomorphic to $\mathbb{R}^k$. Then, $p^{-1}[V] \cap V_y \cong V \cong \mathbb{R}^k$ is an open neighbourhood of y. ■

Given a map $f : (Y,y) \to (X,x)$ between pointed spaces and a covering $p : (\tilde{X},\tilde{x}) \to (X,x)$, when does a lift $\tilde{f} : (Y,y) \to (\tilde{X},\tilde{x})$ exist?

$$\begin{tikzcd} & (\tilde{X},\tilde{x}) \arrow[d, "p"] \\ (Y,y) \arrow[ur, dashed, "\tilde{f}"] \arrow[r, "f"] & (X,x) \end{tikzcd}$$

Lemma 9.3. *If X, $\tilde{X}$, and Y are connected manifolds, then the lift $\tilde{f}$ exists if and only if $f_*\big(\pi_1(Y,y)\big) \subseteq P_*\big(\pi_1(\tilde{X},\tilde{x})\big)$. Moreover, such a lift is unique.*

Two coverings $p : Y \to X$ and $q : Z \to X$ are *isomorphic* if they factor through each other. That is, there exist maps f and g such that

$$p = q \circ f \qquad \text{and} \qquad q = p \circ g$$

This also implies that f and g are inverse, so equivalently, p and q are isomorphic if there exists a homeomorphism $h : Y \to Z$ such that

$$\begin{tikzcd} Y \arrow[rr, "h", "\cong"'] \arrow[dr, "p"'] & & Z \arrow[dl, "q"] \\ & X & \end{tikzcd}$$

commutes.

Example. p_2 is isomorphic to p_{-2} via the homeomorphism $h(z) = z^{-1}$. △

Example. p_2 and p_3 are not isomorphic, as one is a 2-fold covering, and the other is a 3-fold covering. △

Let $p : \tilde{X} \to X$ be a covering of X. A *deck transformation* is a homeomorphism $\tau : \tilde{X} \to \tilde{X}$ such that $p \circ \tau = p$. That is, τ witnesses an automorphism of p. The set of all deck transformations of a cover p is denoted $\mathrm{Deck}(p)$, and has group structure under composition.

Example. The map $z \mapsto -z$ is a deck transformation for p_2. △

Theorem 9.4 (Galois Theory for Covering Spaces)**.** *Let X be a connected manifold. Then, there is a bijection*

$$\{\textit{connected covering spaces of } X\}/\cong \qquad \leftrightarrow \qquad \{\textit{subgroups of } \pi_1(X)\}/\textit{conjugacy}$$

This bijection sends a covering space $p : \tilde{X} \to X$ to the conjugacy class of subgroups $p_*\big(\pi_1(Y)\big) \subseteq \pi_1(X)$. The trivial subgroup corresponds to the *universal cover* $\overline{X} \to X$ – the connected and simply connected covering space of X unique up to isomorphism. Moreover, $\pi_1(X)$ is the group $\mathrm{Aut}_X(\overline{X})$ of isomorphisms, or *deck transformations*, $\overline{X} \to \overline{X}$.

Example. Since p_∞ is a covering space, and $\mathbb{R}$ is connected and simply connected, it is the universal cover. The group of deck transformations is then an infinite cyclic group generated by the map $x \mapsto x+1$. Hence, (if we didn't already know) $\pi_1(S^1) = \mathbb{Z}$. △

The index of the subgroup in $\pi_1(X)$ corresponds to the number of sheets in the covering. So, for example, the n-sheeted covers $p_n : S^1 \to S^1$ correspond to the subgroups $n\mathbb{Z} \leq \mathbb{Z}$.

9.1 Orientations

Let V be a non-zero finite-dimensional real vector space. Recall that two bases define the same orientation if the change of basis transformation from one to the other has positive determinant. This defines an equivalence relation whose two equivalence classes are the two possible orientations of V.

Conversely, we can think of invertible transformations with positive determinant as *orientation-preserving*, and those with negative determinant as *orientation-reversing*.

Example. The determinant of any reflection is -1, so reflections reverse orientation. △

Since manifolds locally appear as finite-dimensional vector spaces, we should expect that orientations can be generalised, at least locally, to manifolds.

Recall that, given a k-manifold M and any point $x \in M$, then the relative homology group at x is given by Corollary 7.4.1 to be:

$$H_(M,M \setminus x) \cong H_{k-1}(S^{k-1}) \cong \mathbb{Z}$$

The choice of generator of $\mathbb{Z}$ is exactly analogous to the choice of basis in a vector space.

Vector Spaces	k-Manifolds
V	$H_k(M,M \setminus x) \cong \mathbb{Z}$
basis	generator
linear transformation	endomorphism of $\mathbb{Z}$
orientation preserving (det > 0)	deg $= 1$
orientation reversing (det < 0)	deg $= -1$

A *local orientation* of M at x is a choice of one of the two generators of $H_k(M,M \setminus x) \cong \mathbb{Z}$.

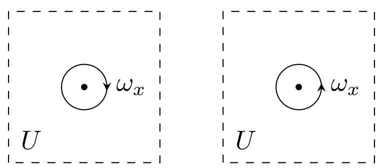
Example. Let M be a 2-manifold, and let $U \ni x$ be an open neighbourhood of x that is homeomorphic to $\mathbb{R}^2$. The long exact sequence for relative homology of the pair $(U, U \setminus x)$ is:

$$\cdots \to H_{n+1}(U) \to H_{n+1}(U,U \setminus x) \to H_n(U \setminus x) \to H_n(U) \to \cdots$$

The open neighbourhood $U \cong \mathbb{R}^2$ is contractible, so the outer terms vanish in degrees $n \geq 1$, and we have an isomorphism in the middle. Also, $U \setminus x$ deformation retracts onto a small circle around x, so,

$$H_2(U,U \setminus x) \cong H_1(U \setminus x) \cong H_1(S^1)$$

Choosing a local orientation ω_x at x therefore amounts to choosing in which direction to traverse this circle:



△

In the above, choosing a local orientation at x also determines the local orientation of every other point y contained in a small neighbourhood around x. In some manifolds, say $\mathbb{R}^2$, this extends to the entire space: we can pick the clockwise or counterclockwise orientation globally.

This is not true on the open Möbius band; if we choose a local orientation and try to "transport" it along a loop around the band, we end up with the opposite orientation after having traversed the band once.

Let $B \subseteq M$ be a subset of a k-manifold. We say that B is a *small open* (resp. *closed*) *ball* if it has an open neighbourhood $U \supseteq B$ homeomorphic to $\mathbb{R}^k$ via, say f, such that $f(B)$ is an open (resp. closed) ball of finite radius.

$$\begin{array}{ccc} U & \xrightarrow[\cong]{f} & \mathbb{R}^k \\ \cup\!\! | & & \cup\!\! | \\ B & \xrightarrow[\cong]{f|_B} & B(x,r) \end{array}$$

The point of this definition is that by excision, then applying the homeomorphisms above:

$$H_k(M,M \setminus B) \cong H_k(U,U \setminus B) \overset{f_*}{\cong} H_k\big(\mathbb{R}^k,\mathbb{R}^k \setminus B(x,r)\big)$$

Then, the long exact sequence in relative homology for the pair $\big(\mathbb{R}^k,\mathbb{R}^k \setminus B(x,r)\big)$ is:

$$\cdots \to H_k(\mathbb{R}^k) \to H_k\big(\mathbb{R}^k,\mathbb{R}^k \setminus B(x,r)\big) \to H_{k-1}\big(\mathbb{R}^k \setminus B(x,r)\big) \to H_{k-1}(\mathbb{R}^k) \to \cdots$$

but $\mathbb{R}^k$ is contractible, so the outer terms vanish, and we have the isomorphism

$$H_k\big(\mathbb{R}^k,\mathbb{R}^k \setminus B(x,r)\big) \cong H_{k-1}\big(\mathbb{R}^k \setminus B(x,r)\big)$$

and finally, $\mathbb{R}^k \setminus B(x,r)$ deformation retracts to the boundary $\partial B(x,r)$, giving

$$H_{k-1}\big(\mathbb{R}^k \setminus B(x,r)\big) \cong H_{k-1}\big(\partial B(x,r)\big) \cong H_{k-1}(S^{k-1}) \cong \mathbb{Z}$$

which is infinite cyclic. Chaining these all together, we have

$$H_k(M,M \setminus B) \cong H_{k-1}\big(\partial B(x,r)\big)$$

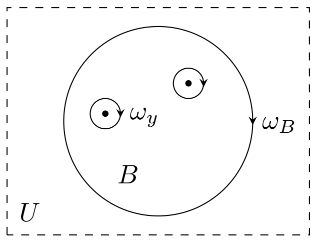
so we can think of a generator here as an orientation of the boundary of B.

Now, for every point $y \in B$, we then get an induced local orientation through the canonical inclusion map

$$H_k(M,M \setminus B) \xrightarrow[\cong]{\mathrm{sp}_y} H_k(M,M \setminus y)$$

because $M \setminus B \subseteq M \setminus y$.

A family of local orientations $(\omega_y)_{y \in B}$ is *consistent* if there is a generator $\omega_B \in H_k(M,M \setminus B)$ such that $\mathrm{sp}_y(\omega_B) = \omega_y$ for all $y \in B$.



An *orientation* of a k-manifold M is a family of local orientations $(\omega_x)_{x \in M}$ which are locally consistent. That is, for all $x \in M$, there exists a small open ball B such that the local orientations $(\omega_y)_{y \in B}$ are consistent.

M is *orientable* if it admits an orientation and is *non-orientable* otherwise.

Example. The k-sphere $M = S^k$ is orientable. For $k = 0$, S^0 is

Choose a generator $\omega \in H_k(S^k)$. Then, for each point $x \in S^k$, the long exact sequence in relative homology for the pair $(S^k,S^k \setminus x)$ is

$$\cdots \to H_k(S^k \setminus x) \to H_k(S^k) \to H_k(S^k,S^k \setminus x) \to H_{k-1}(S^k \setminus x) \to \cdots$$

$S^k \setminus x$ is contractible, so the outer terms vanish, so the map in the centre is an isomorphism:

$$H_k(S^k) \xrightarrow{f_x} H_k(S^k, S^k \setminus x)$$

and hence this map induces local orientations $\omega_x := f_x(\omega)$ at each point $x \in S^k$. These local orientations are also locally consistent, since the map above factors through $H_k(S^k, S^k \setminus B)$ via inclusion for any small open ball B around x. △

We define the *orientation bundle* $\tilde{M}$ to be the set of pairs (x,ω_x), where $x \in M$ and ω_x is a local orientation at x. This set is equipped with the map $\pi : \tilde{M} \to M$ that projects to the first coordinate. We can put a topology on this set using this map.

If $B \subseteq M$ is a small open ball, then we have seen that there are precisely two collections of local orientations $(\omega_y)_{y \in B}$ that are locally consistent in B. In other words,

$$\begin{aligned} \pi^{-1}[B] &= \big(y, \mathrm{sp}_y(\omega_B)\big)_{y \in B} \sqcup \big(y, \mathrm{sp}_y(-\omega_B)\big)_{y \in B} \\ &\cong B_+ \sqcup B_- \end{aligned}$$

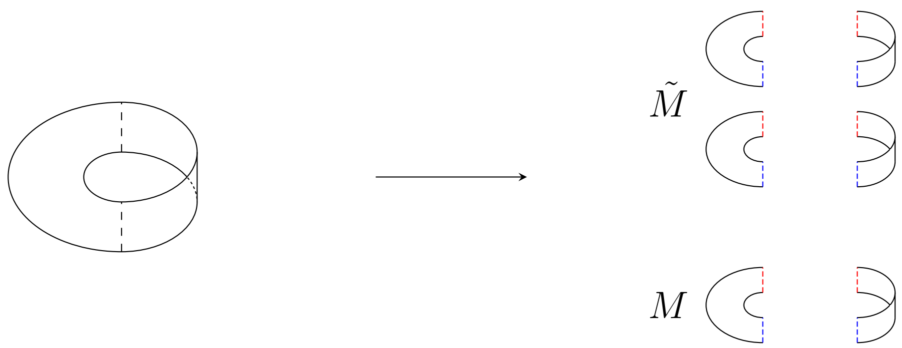
where $B_\pm \overset{\pi}{\cong} B$. We define the topology on $\tilde{M}$ to be generated by the sets $B+$ and B_- for all small open balls $B \subseteq M$.

From this, we have that $\pi : \tilde{M} \to M$ is a 2-fold covering, as, by construction, the preimage of any small open ball consists of two open sets homeomorphic to B under π.

Example. Let M be the open Möbius band

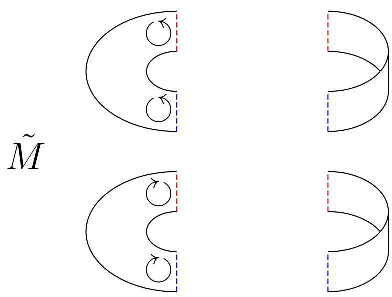
$$M := [0,1] \times (0,1)/\sim$$

where $(0,y) \sim (1,1-y)$ for all $y \in (0,1)$.

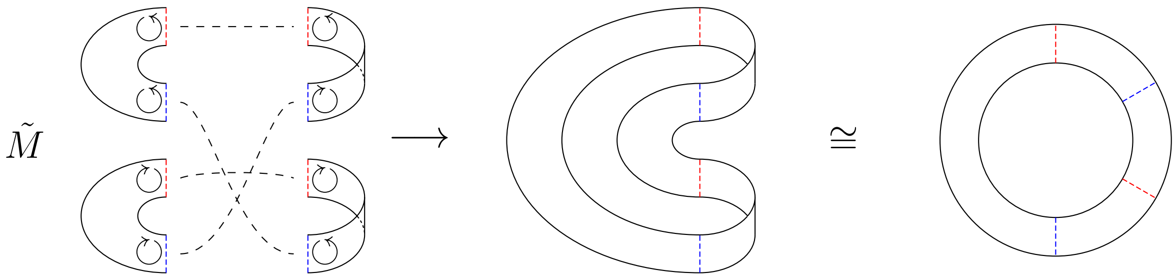


If we cut the Möbius band along these two dashed lines, then we can take the two halves, plus some extra space to overlap, to be two small open balls that jointly cover M. $\tilde{M}$ is a two-fold cover, so we have the setup on the right, where the two copies of M in $\tilde{M}$ have different local orientations.

Pick some orientation in the upper left piece, next to the red boundary. We can transport this orientation down to the blue boundary. Also, the lower left piece must have opposite orientation.



Now, the red boundary at the top must glue to one of the other red boundaries; suppose it glues to the upper right piece. So, this piece shares the same local orientation, but this orientation is reversed by the twist as we reach the blue boundary. Again, the piece below must have opposite local orientations, so altogether, we have:



So $\tilde{M} \cong S^1 \times (0,1)$. △

Lemma 9.5. *Giving an orientation of M is equivalent to giving a continuous section to π.*

Proof. Giving a section $\omega : M \to \tilde{M}$ (not necessarily continuous) amounts to choosing, for each $x \in M$, a local orientation ω_x at x, since the first component must be the identity.

The map ω is continuous if and only if for each small open ball $B \subseteq M$, $\omega^{-1}[B_+]$ and $\omega^{-1}[B_-]$ are open in M, where $B_+ \sqcup B_- := B \sqcup B = \pi^{-1}[B]$.

Since these preimages are disjoint and jointly cover B, this condition is equivalent to $\omega(B) = B_+$ or $\omega(B) = B_-$,which means precisely that the local orientations $(\omega_y)_{y \in B}$ are consistent. ■

Let $x \in M$ and choose a local orientation ω_x. A path in M from x to y has a unique lift to $\tilde{M}$ starting at (x,ω_x) and ending at (y,ω_y) for some ω_y. In other words, this path determins a unique local orientation at y.

Corollary 9.5.1. *If M is a connected manifold, then:*

- *either, $\tilde{M}$ is connected, and M is non-orientable;*
- *or, $\tilde{M} \cong M \sqcup M$, and M admits precisely two orientations.*

Example. We have seen that the orientation bundle of the Möbius band is homeomorphic to $S^1 \times (0,1)$, which is connected. Hence, the Möbius band is not orientable. △

Corollary 9.5.2. *Any simply connected manifold is orientable.*

Theorem 9.6. *Let $k \geq 1$. Then, $\mathbb{RP}^k$ is orientable if and only if k is odd.*

9.2 Surfaces

We define a *surface* here to mean a compact connected 2-manifold. (In particular, a surface is non-empty.)

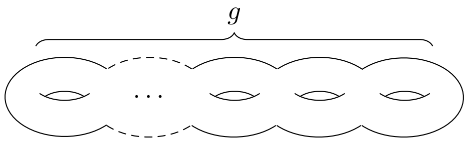
Example. S^2, $\mathbb{T}$, $\mathbb{K}$, and $\mathbb{RP}^2$ are surfaces. △

Let S_1 and S_2 be two surfaces, and let $D_i \subseteq S_i$ be two small closed disks. We can glue $S_1 \setminus D_1^\circ$ and $S_2 \setminus D_2^\circ$ along $\partial D_1 \cong \partial D_2$. The resulting space is called the *connected sum* $S_1 \# S_2$.

The connected sum operation is associative, commutative, and unital on the set of homeomorphism types of surfaces, with the unit being given by the 2-sphere S^2.

Example. The g-holed torus Σ_g can be obtained as the connected sum of g tori:

$$\Sigma_g = \underbrace{\mathbb{T} \# \cdots \# \mathbb{T}}_{g}$$



△

Example. $\mathbb{K} \cong \mathbb{RP}^2 \# \mathbb{RP}^2$. △

Up to homeomorphism, every surface is one of:

(*i*) Σ_g, $g \geq 0$: the integer g is called the *genus* of the surface;

(*ii*) N_h, $h \geq 1$; the integer h is called the *non-orientable genius* of the surface.

Example. The torus $\mathbb{T}$ is of the first type, and has genus 1. The real projective plane $\mathbb{RP}^2$ is of the second type, and has non-orientable genus 1. △

Within each subtype, orientable and non-orientable genus behave well with respect to connected sums. That is,

$$\Sigma_a \# \Sigma_b \cong \Sigma_{a+b} \qquad\qquad N_a \# N_b \cong N_{a+b}$$

However, the connected sum of $\mathbb{T}$ and $\mathbb{RP}^2$ is

$$\mathbb{T} \# \mathbb{RP}^2 \cong N_3$$

Theorem 9.7. *The set of surfaces up to homeomorphism forms a commutative monoid with the connected sum, isomorphic to the monoid with presentation*

$$\langle t,r \mid t + r = 3r \rangle$$

where t represents $\mathbb{T}$, and r represents $\mathbb{RP}^2$.

9.3 Homology and Orientation of Surfaces

Recall that a compact 0-manifold is just a finite discrete set, so the 0th homology group classifies them completely (is a *complete invariant*). Compact 1-manifolds are just finite disjoint unions of circles, so H_0 is also classifies them. H_1 can also distinguish 1-manifolds from 0-manifolds, so (H_0,H_1) is a complete invariant for compact manifolds of dimension at most 1.

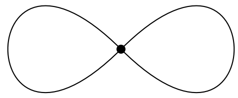
This pattern continues into dimension 2: we will show that (H_0,H_1,H_2) is a complete invariant for compact manifolds of dimension at most 2.

Let $\big((X_\alpha,x_\alpha)\big)_{\alpha\in\Lambda}$ be a collection of pointed spaces. Recall that the *wedge sum* of this collection is the "one-point union" of the spaces, defined as:

$$\bigvee_{\alpha\in\Lambda} (X_\alpha,x_\alpha) \coloneqq \bigsqcup_{\alpha\in\Lambda} X_\alpha / x_\alpha \sim x_\beta$$

That is, the disjoint union of each space with all the basepoints identified.

Example. The wedge sum of two pointed circles is the figure-eight graph:



$S^1 \vee S^1$

$\triangle$

Lemma 9.8. *If each pair in $\big((X_\alpha,x_\alpha)\big)_{\alpha\in\Lambda}$ is a good pair, then*

$$\tilde{H}_n\left(\bigvee_{\alpha\in\Lambda}(X_\alpha,x_\alpha)\right) = \bigoplus_{a\in\Lambda}\tilde{H}_n(X_\alpha)$$

Theorem 9.9. *The homology of the g-holed torus is*

$$H_n(\Sigma_g) = \begin{cases} \mathbb{Z} & n = 0,2 \\ \mathbb{Z}^{2g} & n = 1 \\ 0 & n \geq 3 \end{cases}$$

Corollary 9.9.1. *The surfaces Σ_g, $g \geq 0$, are orientable.*

Theorem 9.10. *The homology of N_h is*

$$H_n(N_h) = \begin{cases} \mathbb{Z} & n = 0 \\ \mathbb{Z}^{h-1} \oplus \mathbb{Z}/2 & n = 1 \\ 0 & n \geq 2 \end{cases}$$

Corollary 9.10.1. *The surfaces N_h, $h > 0$, are non-orientable.*

10 Comparison

At this point, we still have not proved that simplicial homology is an invariant of geometric realisation. That is, that $H_n^\Delta(X)$ is independent from the choice of Δ-complex structure on X.

We will show that the simplicial homology of a space is isomorphic to the singular homology, regardless of the choice of Δ-complex structure.

10.1 Simplicial = Singular

Let X be a topological space with a Δ-complex structure $\big(T,f : |T| \cong X\big)$. Every n-simplex $s \in T$ induces a canonical continuous map $\Delta^n \to X$, which we will also denote by s.

This extends to a homomorphism $\Delta_n(T) \to C_n(X)$ from between simplicial and singular chain groups, and in fact to a chain map $\Delta_\bullet(T) \to C_\bullet(X)$, since the boundary operator is defined in the same way in both chains.

Recall that we "defined" the simplicial homology as:

$$H_n^\Delta(X) \coloneqq H_n(Y)\Big(\coloneqq H_n\big(\Delta_\bullet(T)\big)\Big)$$

Theorem 10.1. *The induced map $H_n^\Delta(X) \to H_n(X)$ is an isomorphism.*

Lemma 10.2. *The canonical homomorphism* $H_n(T^k, T^{k-1}) \to H_n\big(|T^k|, |T^{k-1}|\big)$ *is an isomorphism.*

Corollary 10.2.1. *The simplicial homology* $H_\bullet^\Delta(X)$ *depends only on* X *and not on any* Δ*-complex structure.*

Corollary 10.2.2. *If* X *has a* Δ*-complex structure with simplices in dimensions at most* k*, then* $H_n(X) = 0$ *for all* $n > k$.

10.2 CW Complexes

Recall that a *CW complex* ("*Closure finite*, *Weak topology*") is a topological space* X obtained as follows:

(*i*) Start with a 0-*skeleton* consisting of a disjoint union $X^0 = \bigsqcup_i D_i^0$ of 0-discs (i.e. points), or 0-*cells*.

(*ii*) Given an $(n-1)$-skeleton X^{n-1}, we construct the X^n by gluing a collection of n-*cells* (i.e. n-discs) D_α^n via *attaching maps* $\varphi_\alpha : \partial D_\alpha^n = S_\alpha^{n-1} \to X^{n-1}$:

$$X^n := X^{n-1} \sqcup \textstyle\bigsqcup_\alpha D_\alpha^n / {\sim}$$

where $x \sim \varphi_\alpha(x)$ for all $x \in S_\alpha^{n-1}$.

This recursion then either stops at some finite level n, yielding a CW complex $X := X^n$ of *dimension* n, or continues infinitely, in which case we define $X := \bigcup_{n\in\mathbb{N}} X^n$, with a subspace $U \subseteq X$ being open if and only if $U \cap X^n$ is open in X^n for all n.

For each n-cell D_α^n, we define the *characteristic map* $\Phi_\alpha : D_\alpha^n \to X$ to be the composition

$$D_\alpha^n \hookrightarrow X^{n-1} \sqcup \bigsqcup_\alpha D_\alpha^n \xrightarrow{q} X^n \hookrightarrow X$$

where q is the quotient map induced by φ_α, identifying $x \sim \varphi_\alpha(x)$ for all $x \in S_\alpha^{n-1}$, and the other two maps are the canonical inclusions.

Every Δ-complex is a CW complex. The main difference between the two is that the attaching maps for an n-cell (D^n) in a CW complexes may be *any* continuous map into anywhere in the $(n-1)$-skeleton, while in a Δ-complex, the attaching maps must glue each face of the n-cell (Δ^n) to an $(n-1)$-simplex already in the complex.

In particular, this means that CW complexes may "skip" dimensions, and add no cells in a particular step, but add more cells after. This cannot be the case for a Δ-complex, because the face maps after a skipped step would have no simplices to attach to.

Example. The sphere S^k for $k > 0$ admits a CW complex structure with a single 0-cell, and a single k-cell, where the attaching map is the unique map sending $\partial D^k = S^{k-1}$ to the unique point of the 0-cell. △

Example. A 1-dimensional CW complex is the same thing as a 1-dimensional Δ-complex, and both can be identified with a topological graph. △

Example. Real projective k-space can be constructed as the quotient $\mathbb{RP}^k \cong S^k/(x \sim -x)$ of the k-sphere under the antipodal map. Alternatively, it is the quotient of one of the two hemispheres D^k with antipodal points on the boundary $\partial D^k = S^{k-1}$ identified. But, this boundary is precisely $\mathbb{RP}^{k-1}$, so $\mathbb{RP}^k$ can be obtained by attaching a k-cell to $\mathbb{RP}^k$ along the quotient map $S^{k-1} \to \mathbb{RP}^{k-1}$.

Inductively, it follows that $\mathbb{RP}^k$ has a CW structure with exactly one cell in each dimension $0, 1, \ldots, k$. △

Example. If we continue this process, we can construct the infinite real projective space $\mathbb{RP}^\infty := \bigcup_{k\in\mathbb{N}} \mathbb{RP}^k$ as a CW complex with a single cell in each dimension. △

*As with Δ-complexes, this only describes a CW complex *structure* on a space X, of which there can be many.

10.3 Cellular Homology

We would like to define a homology theory for CW complexes, similar to simplicial homology for Δ-complexes. As before, we can take the group of n-chains $C_n^{\mathrm{CW}}(X)$ to be the free abelian group on the n-cells, but cells in a CW complex may be attached in much more complicated ways than in Δ-complexes, so there isn't an obvious way to define the oriented boundary of an n-cell.

Lemma 10.3. *Let X be a CW complex with n-cells $(D^n_\alpha)_{\alpha\in\Lambda}$, $n > 0$. Then,*

$$H_k(X^n,x^{n-1}) \cong \begin{cases} \bigoplus_{\alpha\in\Lambda}\mathbb{Z} & k = n \\ 0 & k \neq n \end{cases}$$

That is, the nth relative homology of (X^n,X^{n-1}) is the free abelian group on the set of n-cells.

Proof. (X^n,X^{n-1}) is a good pair, so by 7.5,

$$H_k(X^n,X^{n-1}) \cong \tilde{H}(X^n/X^{n-1})$$

The boundaries of the n-cells in X^n are glued into the X^{n-1} skeleton, so in this quotient, their boundaries are all identified together, so

$$X^n/X^{n-1} \cong \bigvee_{\alpha\in\Lambda} D^n_\alpha/\partial D^n_\alpha \cong \bigvee_{\alpha\in\Lambda} S^n$$

Relative homology splits across wedge sums, so

$$\tilde{H}\left(\bigvee_{\alpha\in\Lambda} S^n\right) \cong \bigoplus_{\alpha\in\Lambda}\tilde{H}_k(S^n) \cong \begin{cases} \bigoplus_{\alpha\in\Lambda}\mathbb{Z} & k = n \\ 0 & k \neq n \end{cases}$$

■

We can describe this isomorphism explicitly as follows. Choose a homeomorphism $f : \Delta^n \xrightarrow{\cong} D^n$. Then, we have a continuous map given by the composition

$$\Delta \xrightarrow{f} D^n \xrightarrow{\Phi_\alpha} X^n$$

which is in fact a relative cycle for the pair (X^n,X^{n-1}), and its relative homology class generates the copy of $\mathbb{Z}$ corresponding to D^n_α.

Lemma 10.4. *Let X be a CW complex. Then,*

(i) $H_n(X^k) = 0$ *for all* $n > k$.

In particular, the homology of X vanishes in all degrees $n > \dim(X)$;

(ii) $H_n(X^k) \cong H_n(X)$ *for all* $n < k$.

More specifically, the map $\iota_* : H_n(X^k) \to H_n(X)$ induced by the inclusion $\iota : X^k \hookrightarrow X$ is an isomorphism if $n < k$, and is surjective if $n = k$.

Proof.

(i) Consider the long exact sequence for relative homology of the pair (X^k,X^{k-1}):

$$\cdots \to H_{n+1}(X^k,X^{k-1}) \to H_n(X^{k-1}) \to H_n(X^k) \to H_n(X^k,X^{k-1}) \to \cdots$$

From the previous lemma, the left relative homology group vanishes for $n+1 \neq k$, and the right group vanishes for $n \neq k$. So, both outer terms vanish in degrees $n \neq k,k-1$, and we have an isomorphism in the middle. So, if $n > k$,

$$H_n(X^k) \cong H_n(X^{k-1}) \cong \cdots \cong H_n(X^0) \cong 0$$

(*ii*) Suppose X has finite dimension d. If $n < k$, then we have

$$H_n(X^k) \cong H_n(X^{k+1}) \cong H_n(X^{k+2}) \cong \cdots \cong H_n(X^d) = H_n(X)$$

so $H_n(X^k) \cong H_n(X)$.

Otherwise, if $n = k$, then only the right group vanishes, and we only have a surjection in this degree, so

$$H_n(X^k) \twoheadrightarrow H_n(X^{k+1}) \cong H_n(X^{k+2}) \cong \cdots \cong H_n(X^d) = H_n(X)$$

so $H_n(X^k) \twoheadrightarrow H_n(X)$.

If X is infinite-dimensional, the proof is complicated and is omitted.

■

We can now describe cellular homology. We will take the relative homology groups

$$H_{n+1}(X^{n+1},X^n) \longrightarrow H_n(X^n,X^{n-1}) \longrightarrow H_{n-1}(X^{n-1},X^{n-2})$$

to be the chain groups. By Theorem 10.3, these are all free abelian on the set of cells of the matching dimension. The task is now to construct these differentials.

We construct the long exact sequences for the three chain groups, and they fit together into a diagram:

$$\begin{tikzcd}
 & & & 0 & \\
0 \arrow[dr] & & H_n(X^{n+1}) \arrow[ur] & & \\
 & H_n(X^n) \arrow[ur] \arrow[dr, "\beta"] & & & \\
H_{n+1}(X^{n+1},X^n) \arrow[ur, "\alpha"] \arrow[rr, "d_{n+1}"] & & H_n(X^n,X^{n-1}) \arrow[rr, "d_n"] \arrow[dr, "\gamma"'] & & H_{n-1}(X^{n-1},X^{n-2}) \\
 & & & H_{n-1}(X^{n-1}) \arrow[ur, "\delta"'] & \\
 & & 0 \arrow[ur] & &
\end{tikzcd}$$

The leftmost zero is $H_n(X^{n-1})$, which vanishes by Theorem 10.4; the upper zero is $H_n(X^{n+1},X^n)$, which vanishes by Theorem 10.3; and the lower zero is $H_{n-1}(X^{n-2})$, which vanishes by Theorem 10.4.

Also, note that the group at the top $H_n(X^{n+1})$ is isomorphic to $H_n(X)$ via Theorem 10.4.

We define the differentials to be the compositions

$$d_{n+1} \coloneqq \beta \circ \alpha \qquad\qquad d_n \coloneqq \delta \circ \gamma$$

This defines a chain complex, since the composition $d_n \circ d_{n+1}$ factors through β and γ. These are consecutive maps in a long exact sequence, so their composition is zero.

Let X be a CW complex. We define the *cellular chain complex* $C_\bullet^{\mathrm{CW}}(X)$ by

$$\cdots \to C_{n+1}^{\mathrm{CW}} = H_{n+1}(X^{n+1},X^n) \xrightarrow{d\_n+1} C_n^{\mathrm{CW}}(X) = H_n(X^n,X^{n-1}) \xrightarrow{d_n} \cdots$$

The *cellular homology groups* are the homology of this chain complex:

$$H_n^{\mathrm{CW}}(X) \coloneqq H_n\big(C_\bullet^{\mathrm{CW}}(X)\big)$$

Lemma 10.5. *The cellular homology group $H_n^{CW}(X)$ is canonically isomorphic to the singular homology group $H_n(X)$.*

Proof.

$$\begin{aligned}
H_n^{\mathrm{CW}}(X) &:= \frac{\ker(d_n)}{\operatorname{im}(d_{n+1})} \\
&\cong \frac{\ker(\gamma)}{\operatorname{im}(d_{n+1})} && \text{[by injectivity of } \delta] \\
&\cong \frac{\operatorname{im}(\beta)}{\operatorname{im}(d_{n+1})} && \text{[exactness at } C_n^{\mathrm{CW}}(X)] \\
&\cong \frac{\operatorname{im}(\beta)}{\operatorname{im}(\beta \circ \alpha)} \\
&\cong \frac{H_n(X^n)}{\operatorname{im}(\beta \circ \alpha)} && \text{[by injectivity of } \beta] \\
&\cong \frac{H_n(X^n)}{\operatorname{im}(\alpha)} \\
&\cong \operatorname{coker}(\alpha) \\
&\cong H_n(X^{n+1}) \\
&\cong H_n(X)
\end{aligned}$$

■

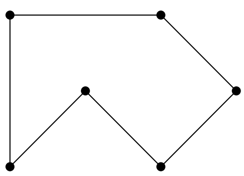
Corollary 10.5.1. *For any CW complex X, there are canonical isomorphismss $H_n^{CW}(X) \cong H_n(X)$.*

Theorem 10.6. *The boundary operator for cellular homology is given by*

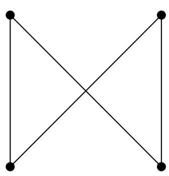
- *in degree $n = 1$:*
$$d_1(D_\alpha^1) = \varphi$$
- *in degrees $n > 1$:*
$$d_n(D_\alpha^n) = \sum_\beta d_{\alpha\beta} D_\beta^{n-1}$$
where
$$d_{\alpha\beta} = \deg\left(\Delta_{\alpha\beta} : S_\alpha^{n-1} \xrightarrow{\varphi_\alpha} X^{n-1} \xrightarrow{\pi_\beta} S_\beta^{n-1}\right)$$

11 The Euler Characteristic

A *plane graph* is a finite 1-dimensional CW complex embedded in the real plane $\mathbb{R}^2$. Equivalently, it is a finite graph in the plane in which the edges do not cross.



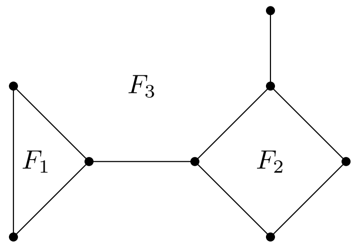
a plane graph



not a plane graph

Note that this is distinct from the notion of a *planar graph* in graph theory. Some finite graphs, for instance, the complete graph K_5 or the complete bipartite graph $K_{3,3}$, do not admit an embedding into $\mathbb{R}^2$, so they are *nonplanar*. The example on the right above *does* admit an embedding into $\mathbb{R}^2$ as a square, so it is *planar*, but the particular embedding shown is not a *plane* graph.

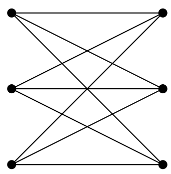
A *face* of a plane graph $\mathscr{G}$ is a connected component of $\mathbb{R}^2 \setminus \mathscr{G}$:



Theorem 11.1 (Euler)**.** *For a plane graph with v vertices, e edges, and f faces,*

$$v - e + f = 2$$

Example. The complete bipartite graph



has $v = 6$ vertices and $e = 9$ edges. Suppose that $K_{3,3}$ admits an embedding into $\mathbb{R}^2$. Because it is bipartite, all cycles have even length, so every face has at least 4 edges. So, $4f \leq 2e = 18$, or $f \leq 4$. Hence

$$v - e + f \leq -3 + 4 = 1$$

contradicting Euler's formula, so $K_{3,3}$ is nonplanar. △

By taking the one-point compactification of $\mathbb{R}^2$, a planar graph yields a CW complex structure on the $\mathbb{R}^2 \cup \{\infty\} \cong S^2$ with v many 0-cells, e 1-cells, and f 2-cells.

This result is non-trivial to prove (compare with the Jordan curve theorem), and is a special case of the Schoenflies' theorem, which we will assume without proof.

Given a topological space X that admits a finite CW complex structure, we define the *Euler characteristic* to be the alternating sum

$$\chi(X) := \sum_{n \in \mathbb{N}} (-1)^n \big|\{n\text{-cells in } X\}\big|$$

of the number of n-cells in X.

Theorem 11.2. *The Euler characteristic is independent of choice of CW complex structure.*

Let A be a finitely generated abelian group. It decomposes as $A = F \oplus T$, the direct sum of the torsion-free part $T \cong \mathbb{Z}^r$, which is necessarily free abelian, and a finite abelian group F. The integer r is well-defined and is called the *rank* of A, denoted $\mathrm{rk}_{\mathbb{Z}}(A)$. (See § 1.7.3.)

Alternatively, this rank is given by the dimension of the $\mathbb{Q}$-vector space $A \otimes_{\mathbb{Z}} \mathbb{Q}$ since $\mathbb{Z}^r \otimes_{\mathbb{Z}} \mathbb{Q} = \mathbb{Q}^r$, and $F \otimes_{\mathbb{Z}} \mathbb{Q} = 0$.

Theorem 11.3. *For a short exact sequence $0 \to A_1 \to A_2 \to A_3 \to 0$ of finitely generated abelian groups,*

$$\mathrm{rk}_{\mathbb{Z}}(A_1) - \mathrm{rk}_{\mathbb{Z}}(A_2) + \mathrm{rk}_{\mathbb{Z}}(A_3) = 0$$

Corollary 11.3.1. *Let $C_\bullet$ be a chain complex with finitely many non-zero terms, all of which are finitely generated abelian groups. Then,*

$$\sum_n (-1)^n \operatorname{rk}_{\mathbb{Z}}(C_n) = \sum_n (-1)^n \operatorname{rk}_{\mathbb{Z}}\big(H_n(C_\bullet)\big)$$

Let X be a space with only finitely many non-zero homology groups, all of which are finitely generated abelian groups. Then, its Euler characteristic is defined as

$$\chi(X) \coloneqq \sum_{n\in\mathbb{N}} (-1)^n \operatorname{rk}_{\mathbb{Z}}\big(H_n(X)\big)$$

Example.

$$\chi(S^k) = \begin{cases} 2 & k \text{ even} \\ 0 & k \text{ odd} \end{cases}$$

△

Theorem 11.4. *Let $X = U \cup V$ and either*

- *X is a CW complex and U,V are subcomplexes;*
- *or X is any topological space and $U,V \subseteq X$ are open subsets.*

Then, if $\chi(U)$, $\chi(V)$, and $\chi(U \cap V)$ are all defined, then so is $\chi(X)$, and it is given by

$$\chi(X) = \chi(U) + \chi(V) - \chi(U \cap V)$$

Corollary 11.4.1. *We have*

$$\chi(\Sigma_g) = 2 - 2g$$
$$\chi(N_h) = 2 - h$$

The Euler characteristic of a surface can therefore attain any integer less than or equal to 2, with equality only in the case of the 2-sphere.

Every even non-positive integer is the Euler characteristic of precisely two surfaces, one orientable and one non-orientable. In particular, surfaces are completely classified by

- whether they are orientable or not, and
- their Euler characteristic.

Theorem 11.5. *Let X and Y be finite CW complexes. Then so is $X \times Y$,*

$$\chi(X \times Y) = \chi(X) \cdot \chi(Y)$$

Theorem 11.6. *Let $p : Y \to X$ be an n-fold covering, and suppose that X is a finite CW complex. Then, Y is also a finite CW complex, and we have*

$$\chi(Y) = n \cdot \chi(X)$$

12 Results from Homological Algebra

12.1 Common Exact Sequences

- If
$$0 \to A \xrightarrow{f} B \to 0$$
is exact, then $A \cong B$; exactness at A gives $\ker(f) = \operatorname{im}(0) = 0$, so f is injective; and exactness at B gives $\operatorname{im}(f) = \ker(0) = B$, so f is surjective.
- If
$$0 \xrightarrow{f} A \xrightarrow{g} 0$$
is exact, then $A \cong 0$, as $0 = \operatorname{im}(f) = \ker(g) = A$.
- If
$$0 \to A \to B \to C \to 0$$
is exact and C is free abelian (or projective), then $B \cong A \oplus C$.
- If
$$0 \to A \xrightarrow{a} B \xrightarrow{f} C \xrightarrow{b} D \to 0$$
is exact, then $A \cong \ker(f)$ and $D \cong \operatorname{coker}(f)$: exactness at A gives $\ker(a) = \operatorname{im}(0) = 0$, so a is injective and hence $A \cong \operatorname{im}(a)$; by exactness at B, $A \cong \operatorname{im}(a) = \ker(f)$.

12.2 Splitting Lemma

Lemma 12.1. *Given a short exact sequence*
$$0 \to A \xrightarrow{q} B \xrightarrow{r} C \to 0$$
the following statements are all equivalent:

(*i*) ***Left split****: There exists a morphism $t : B \to A$ such that $t \circ q = \mathrm{id}_A$;*

(*ii*) ***Right split****: There exists a morphism $u : C \to B$ such that $r \circ u = \mathrm{id}_C$;*

(*ii*) *There is an isomorphism $h : B \to A \oplus C$, such that $h \circ q = \iota_1 : A \hookrightarrow A \oplus C$ is the canonical inclusion mapping, and $r \circ h^{-1} = \pi_2 : A \oplus C \to C$ is the canonical projection mapping.*

If any of these statements hold, then the sequence is called a *split exact sequence*, or the sequence is said to *split*.

12.3 Five Lemma

Lemma 12.2 (Five Lemma)**.** *For the following commutative diagram,*

$$\begin{tikzcd}
A \arrow[r, "a"] \arrow[d, "\alpha"] & B \arrow[r, "b"] \arrow[d, "\beta"] & C \arrow[r, "c"] \arrow[d, "\gamma"] & D \arrow[r, "d"] \arrow[d, "\delta"] & E \arrow[d, "\varepsilon"] \\
A' \arrow[r, "a'"] & B' \arrow[r, "b'"] & C' \arrow[r, "c'"] & D' \arrow[r, "d'"] & E'
\end{tikzcd}$$

if the two rows are exact, β and δ are isomorphisms; α is an epimorphism; and ε is a monomorphism, then γ is an isomorphism.

12.4 Nine Lemma

Lemma 12.3 (Nine Lemma)**.** *In the following commutative diagram,*

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & A_1 \arrow[r] \arrow[d] & B_1 \arrow[r] \arrow[d] & C_1 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A_2 \arrow[r] \arrow[d] & B_2 \arrow[r] \arrow[d] & C_2 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A_3 \arrow[r] \arrow[d] & B_3 \arrow[r] \arrow[d] & C_3 \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

if all columns and the lower two rows are exact, then the top row is exact as well. Similarly, if all columns and the upper two rows are exact, then the bottom row is exact as well.

The diagram is symmetric about the diagonal, so rows and columns may be interchanged in the above as well.

12.5 Snake Lemma

Lemma 12.4 (Snake Lemma)**.** *In the following commutative diagram,*

$$\begin{tikzcd}
 & A \arrow[r, "f"] \arrow[d, "a"] & B \arrow[r, "g"] \arrow[d, "b"] & C \arrow[r] \arrow[d, "c"] & 0 \\
0 \arrow[r] & A' \arrow[r, "f'"] & B' \arrow[r, "g'"] & C' &
\end{tikzcd}$$

if the rows are exact, then there is a connecting homomorphism $\partial : \ker(c) \to \operatorname{coker}(a)$ *and an exact sequence*

$$\ker(a) \to \ker(b) \to \ker(c) \xrightarrow{\partial} \operatorname{coker}(a) \to \operatorname{coker}(b) \to \operatorname{coker}(c)$$

Moreover, if f is a monomorphism, then so is $\ker(a) \to \ker(b)$*; and if g' is an epimorphism, then so is* $\operatorname{coker}(b) \to \operatorname{coker}(c)$.